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Non-linear wave-number interaction in near-critical two-dimensional flows

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This paper deals with a system of equations which includes as special cases the equations governing such hydrodynamic stability problems as the Taylor problem, the Bénard problem, and the stability of plane parallel flow. A non-linear analysis is made of disturbances to a basic flow. The basic flow depends on a single co-ordinate η. The disturbances that are considered are represented as a superposition of many functions each of which is periodic in a co-ordinate ξ normal to η and is independent of the third co-ordinate direction. The paper considers problems in which the disturbance energy is initially concentrated in a denumerable set of 'most dangerous' modes whose wave-numbers are close to the critical wave-number selected by linear stability theory. It is a major result of the analysis that this concentration persists as time passes. Because of this the problem can be reduced to the study of a single non-linear partial differential equation for a special Fourier transform of the modal amplitudes. It is a striking feature of the present work that the study of a wide class of problems reduces to the study of this single fundamental equation which does not essentially depend on the specific forms of the operators in the original system of governing equations. Certain general conclusions are drawn from this equation, for example for some problems there exist multi-modal steady solutions which are a combination of a number of modes with different spatial periods. (Whether any such solutions are stable remains an open question.) It is also shown in other circumstances that there are solutions (at least for some interval of time) which are non-linear travelling waves whose kinematic behaviour can be clarified by the concept of group speed.

1. Introduction

We begin this introductory section with a brief exposition, in general terms, of the class of problems under consideration and mention some of the results we have achieved. We then discuss more fully the issues with which we are concerned We conclude the introduction with an outline of the rest of the paper.

As exemplified in the Taylor and Bénard problems (see below) the instability of certain simple flows is followed by the appearance of new equilibrium flows which seem to be spatially periodic. This paper is a study of non-linear interactions which include, as an important special case, those which lead to such equilibrium flows. The analysis deals with situations wherein the dependent variables (e.g. three velocity components, pressure, density) are functions of two spatial variables and the time. When functions have this type of spatial dependence they will be referred to as two-dimensional.

Previous studies by various workers (see references below) have dealt with the growth of a two-dimensional infinitesimal disturbance which is periodic in one spatial co-ordinate with wave-number k, the generation of the harmonics (wave-numbers $2k, 3k, \ldots$) of the fundamental mode of this disturbance, the resulting change in the mean motion (zero wave-number), and the equilibration process by which the amplitudes of the fundamental and its harmonics reach final values (which may be steady or may be periodic in time). Further studies have dealt with the stability of these spatially periodic flows to small two-dimensional perturbations and have shown that certain classes of such flows are stable. Three-dimensional problems have also been treated by various authors but such problems will not be considered here.

The stability analyses just mentioned may be regarded as demonstrations that if a certain spatially periodic fundamental mode and its harmonics appear dominantly in the initial disturbance then they will appear exclusively in the final state. The present paper treats multi-modal *initial conditions* in which a combination of any or all of the 'most dangerous modes' is dominant.† We show that no new modes become dominant as time passes, so that the energy spectrum remains concentrated near the critical wave-number. As a consequence of this *permanence of energy concentration* the study of interactions of modes of different wave-numbers can, for a large class of problems, be reduced to the study of a certain particular partial differential equation. This equation contains only two parameters, the growth rate of linear theory and the Landau constant of the simplest non-linear theory.

Several general conclusions can be drawn for this class of problems. For example, in the Taylor and Bénard problems non-linear effects are known to be stabilizing in the sense that they bring to equilibrium spatially periodic disturbances (fundamental plus harmonics) which grow exponentially according to linear stability theory; we show that the non-linear effects remain stabilizing even for the multi-modal flows which result when all the most dangerous modes are allowed to interact. Another general conclusion concerns the effect of non-linearities when linear stability theory predicts that neutral spatially periodic disturbances take the form of travelling waves. We show in this case that if non-linear terms are stabilizing then, for a considerable time span at least, one can anticipate that small linearly unstable perturbations will tend to form a spatially periodic wave train, modulated by an envelope travelling at the 'critical' group speed.

† 'Most dangerous modes' are periodic disturbances which grow, or decay very slowly, according to linear theory. A precise definition will be given later.

The calculations in this paper deal with flows (or other physical phenomena) which are governed by equations which will be discussed in §2. At this point, rather than attempting to characterize precisely the class of mathematical problems to which our analysis applies, we prefer to list three fluid-mechanical problems each of which is a special case of the general problem we shall discuss.

Consider, then, the following three prototypical fluid instability phenomena: (*a*) Flow between a long stationary outer cylinder and a concentric rotating inner cylinder takes place along circular streamlines (Couette flow) if a suitable dimensionless measure of the inner rotation speed (the Taylor number) is small enough. But Taylor vortices spaced periodically in the axial direction appear when the Taylor number exceeds a critical value. (*b*) A horizontal layer of fluid heated from below remains quiescent if a dimensionless imposed temperature gradient (the Rayleigh number) remains small enough, but convects in spatially periodic Bénard cells if the Rayleigh number exceeds a critical value. (*c*) Fluid forced by a pressure gradient to move between parallel planes takes up the parabolic velocity profile of plane Poiseuille flow if its dimensionless maximum speed (a Reynolds number) is small enough, but amplified disturbances which are periodic in the downstream direction (Tollmien–Schlichting waves) develop if the Reynolds number exceeds a critical value.

Each of these problems permits a basic solution (denoted here by $\mathbf{\Phi}_0$) to its governing equations which depends on a single co-ordinate η, where η measures distance normal to the bounding surfaces. In each problem it can be shown that in studying the onset of instability it is sufficient to restrict consideration to a perturbation which depends on only one of the co-ordinate directions normal to η. The corresponding co-ordinate will be denoted by ξ; ξ is the axial co-ordinate in the Taylor problem (*a*), any horizontal co-ordinate in the Bénard problem (*b*), and the downstream co-ordinate in the plane Poiseuille problem (*c*).

Stability theory is the study of perturbations $\mathbf{\Phi}'$, where $\mathbf{\Phi} \equiv \mathbf{\Phi}_0 + \mathbf{\Phi}'$. In linear theory, only terms linear in $\mathbf{\Phi}'$ are retained. In treating initial values of $\mathbf{\Phi}$ which are periodic in ξ with wave-number k, normal mode solutions of the form

$$\mathbf{\Phi}'(\xi, \eta, t) = \epsilon \boldsymbol{\phi}^{(k)}(\eta) \exp\left[-ik\xi - \mu^{(k)}(R)\, t\right] \tag{1.1}$$

are assumed. (The small parameter ϵ, defined in (1.5), is inserted here to obtain conformity with later notation.) The governing equations for the $\boldsymbol{\phi}^{(k)}$ involve a parameter R: the Taylor, Rayleigh and Reynolds numbers in problems (*a*), (*b*) and (*c*) respectively. These equations (and boundary conditions) are linear and homogeneous, and form an eigenvalue problem for the $\mu^{(k)}$. We assume that there exists a denumerably infinite set of eigenvalues $\mu_m^{(k)}(R)$ and corresponding eigenfunctions $\boldsymbol{\phi}_m^{(k)}$ such that the eigenvalues can be ordered:

$$\operatorname{Re} \mu_{m+1}^{(k)} \geqslant \operatorname{Re} \mu_m^{(k)}, \quad m = 0, 1, \ldots.$$

For each k, the critical value of R, $R_c(k)$, is defined by

$$\tau_0^{(k)}[R_c(k)] = 0, \quad \text{where} \quad \tau_0^{(k)} = \operatorname{Re} \mu_0^{(k)}. \tag{1.2}$$

The neutral curve is given by $R = R_c(k)$. On it $\tau_0^{(k)}(R) = 0$. In the problems under consideration there exists a minimum value of $R_c(k)$, R_c, and a corresponding critical wave-number k_c with the following properties.

$$\left.\begin{array}{ll} \text{For} \quad R < R_c\text{:} & \tau_0^{(k)} > 0. \\ \text{For} \quad R = R_c\text{:} & \tau_0^{(k)} > 0 \quad \text{when} \quad k \neq k_c; \quad \tau_0^{(k_c)} = 0. \\ \text{For} \quad R > R_c\text{:} & \begin{cases} \tau_0^{(k)} > 0 \quad \text{when} \quad k < k_1(R), \quad k > k_2(R); \\ \tau_0^{(k)} < 0 \quad \text{when} \quad k_1(R) < k < k_2(R). \end{cases} \end{array}\right\} \tag{1.3}$$

(This is depicted in figure 1(*a*).) Thus linear theory predicts instability of the basic flow when R exceeds R_c. For a given value of $R > R_c$, all modes with wave-numbers between k_1 and k_2 are predicted to grow exponentially in time.

The first step in developing a non-linear stability theory (Stuart 1960; Watson 1960) is to consider the modification of the exponential growth of a single periodic disturbance having wave-number k. Solutions can be found of the form

$$\mathbf{\Phi}'(\xi, \eta, t) = \epsilon A_0^{(k)}(t)\, \boldsymbol{\phi}_0^{(k)}(\eta)\, e^{-ik\xi} + \text{conjugate} + \text{higher-order harmonics}, \tag{1.4}$$

where ϵ, defined by

$$[\epsilon(k, R)]^2 = |\tau_0^{(k)}(R)|, \tag{1.5}$$

is a small parameter. It is not difficult to show by expanding $\tau_0^{(k)}$ about $R_c(k)$ that for any fixed k

$$|R - R_c(k)| = O[\epsilon^2(k, R)]. \tag{1.6a}$$

In particular for $k = k_c$ we have

$$|R - R_c| = O[\epsilon^2(k_c, R)] = O(\epsilon_c^2). \tag{1.6b}$$

Stuart and Watson have shown that the function $A_0^{(k)}(t)$, which gives the amplitude of the fundamental mode of the disturbance with spatial wave-number k, satisfies

$$d|A_0^{(k)}|^2/dt + 2\tau_0^{(k)}|A_0^{(k)}|^2 = -2|\tau_0^{(k)}| \operatorname{Re}\beta^{(k)}|A_0^{(k)}|^2 + \text{higher-order terms}, \tag{1.7}$$

where the *Landau constant* $\beta^{(k)}$ is determined by a solvability condition. (Also see Eckhaus 1965, chapter 7.)

If $\operatorname{Re}\beta^{(k)} < 0$ as is the case in the Taylor and Bénard problems (Davey 1962; DiPrima 1967; Segel 1962), then a periodic two-dimensional equilibrium flow of wave-number k exists under *super-critical conditions* [where $R > R_c(k)$] for each $k \in (k_1, k_2)$. This flow will be stable to two-dimensional infinitesimal perturbations of other wave-numbers if and only if its wave-number lies in a subinterval (k_1^*, k_2^*) as depicted in figure 1(*b*) (Eckhaus 1965; Kogelman & DiPrima 1970).

If $\operatorname{Re}\beta^{(k)} > 0$, as appears to be the case for some values of k in plane Poiseuille flow (Reynolds & Potter 1967; Pekeris & Shkoller 1967) then it can be seen that sufficiently large periodic perturbations to the basic flow will magnify even though small disturbances decay.

The above discussion has been concerned with an initial disturbance composed of a fundamental mode with wave-number k and its harmonics of smaller order in magnitude. Now suppose that we have an initial perturbation composed principally of a denumerable set of modes with wave-numbers $k+n\Delta k$ within $O(\epsilon_c)$ of k_c and Fourier coefficients $\mathbf{\Phi}_{(k+n\Delta k)}(\eta, t)$, $n = 0, \pm 1, \pm 2, \ldots$. There will also be harmonics of smaller magnitude.

What then? For a single mode the non-linear terms are stabilizing in the Taylor and Bénard problems, but do they remain so when relatively strong interactions between modes are possible? Are there any solutions involving a combination of wave-numbers? If so, are they stable? The answers to such questions are not

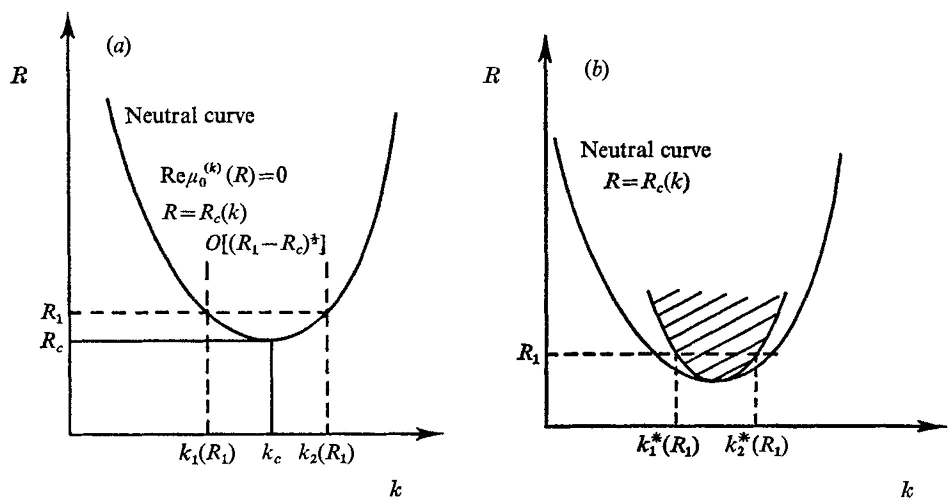

FIGURE 1. Previously known information about spatially periodic perturbations to a basic flow. (*a*) Results of linear theory. The parameter R characterizes the basic flow, k is the wave-number of the perturbation, and $\mu_0^{(k)}(R)$ is the growth rate according to linear theory. The neutral curve separates points (k, R) corresponding to exponentially growing perturbations [Re $\mu_0^{(k)}(R) < 0$] from points corresponding to exponentially decaying perturbations [Re $\mu_0^{(k)}(R) > 0$]. Growth or decay may occur in an oscillatory fashion. (*b*) Results of non-linear theory. Suppose that $R-R_c$ is small. If the real part of the Landau constant β_c (see text) is positive, spatially periodic perturbations corresponding to points (k, R) above the neutral curve lead to new supercritical equilibrium flows. The requirement that these flows be stable to two-dimensional perturbations (see text for exact definition) restricts the corresponding points (k, R) to the shaded region.

easy to determine since they require a study of closely coupled interactions governed by non-linear partial differential equations. In this paper we shall show that for a large class of problems these questions can be answered by studying one specific system of non-linear ordinary differential equations for amplitude functions $A_n(t)$ where

$$\mathbf{\Phi}_{(k+n\Delta k)}(\eta, t) = \epsilon A_n(t)\, \boldsymbol{\phi}_0^{(k+n\Delta k)}(\eta) + \text{higher-order terms.}$$

More advantageously, we shall show that one can study a single non-linear partial differential equation for a special Fourier transform of the A_n's.

A difficulty which had to be overcome is this. Suppose, for example, that the initial perturbation consists solely on two modes with wave-numbers k_c and $k_c+\Delta k$, say, such that $k_1 < k_c < k_c+\Delta k < k_2$ and such that each mode is initially of magnitude $O(\epsilon_c)$. The first non-linear interaction of these modes produces harmonics with wave-numbers 0, Δk, $2k_c$, $2k_c+\Delta k$, $2k_c+2\Delta k$, each of whose magnitude is $O(\epsilon_c^2)$ Next, the interaction of the forced mode $(2k_c)$ of magnitude $O(\epsilon_c^2)$ with the fundamental $(k_c+\Delta k)$ of magnitude $O(\epsilon_c)$ would produce a mode with wave-number $k_c-\Delta k$ and magnitude $O(\epsilon_c^3)$. Now, and this is of primary interest, if $\Delta k = O(\epsilon_c)$ or smaller, the wave-number $k_c-\Delta k$ is close enough to k_c so that the corresponding mode would 'have a life of its own' and as a consequence would grow rapidly. This means that the amplitude equations must involve $A_{-1}(t)$ (even though the corresponding mode was initially absent) in order that the appropriate solvability condition be satisfied. (For this reason the two-mode interaction calculations of Segel (1962) are suggestive at best.) If the initial state consists solely of modes proportional to k_c, $k_c+\Delta k$ and $k_c-\Delta k$ then the same reasoning shows that modes of wave-number $k_c+2\Delta k$ and $k_c-2\Delta k$ will rapidly appear, etc. Thus, as soon as two or more modes with near-critical wave-numbers are initially present at comparable magnitudes, energy spreads rapidly to adjacent modes. At first sight this seems to negate the possibility of a study of wave-number interaction short of a simultaneous consideration of all wave-numbers differing from k_c by some multiple of Δk. However, by a fairly delicate sequence of order of magnitude estimates, we shall show that if energy is initially concentrated in Fourier components with wave-numbers k where $|k-k_c| = O(\epsilon_c)$, then spreading will not continue beyond this neighbourhood.

As mentioned earlier our detailed calculations are carried out for a system of governing equations which has considerable generality. For this reason the calculations apply not only to all three hydrodynamic instability problems mentioned above, but in addition, they apply with little or no modification to a number of other hydrodynamic stability problems and to stability problems in different physical contexts. Examples of the last category are the formation of uneven solute distribution patterns in the freezing of metallic alloys (Wollkind & Segel 1970) and the formation of Abrikosov mixed states in superconducting materials exposed to a magnetic field, once the correct time-dependent version of the Ginzburg–Landau equations is formulated (Odeh 1968).

From a mathematical point of view, this paper is a step in the study of systems of non-linear partial differential equations in the vicinity of a bifurcation point. Most of the rigorous work in this area deals with establishment of sufficient conditions for the existence of a bifurcation point for time-independent equations, and the elucidation of the steady solutions which exist when the governing parameter exceeds the critical value at which bifurcation sets in. Considerable formal work has dealt with the problem of associating sets of possible initial conditions with the steady solution to which they tend as time increases. For an introduction to the literature see the survey by Görtler & Velte (1967), the collection of papers edited by Keller & Antman (1968), and the conference proceedings edited by Liepholz (1971). The last-mentioned reference contains Eckhaus (1970), a paper that briefly summarizes a portion of the present analysis in a discussion of recent developments concerning the stability of periodic flows.

We turn in the next section to a specification of the class of problems which we shall consider. In order to establish notation and to lay the foundation for succeeding generalizations it is necessary to recapitulate the results of linear stability theory and of the non-linear stability theory which describes the behaviour of a periodic perturbation.

In §3 we formulate the problem of the interaction of a large number of initially comparable modes with wave-numbers near k_c, and we outline the extensive calculations which reduce the problem to the non-linear system of ordinary differential equations (3.11).

In §4 we present an order of magnitude analysis which shows that if solutions to (3.11) are bounded then it is consistent to assume, as we have, that modes of wave-number k have relatively small magnitudes unless k is close to k_c.

In §5 we complete the consistency argument by proving that the solutions of (3.11) are indeed bounded. We derive a single partial differential equation for a special Fourier transform of the modal amplitudes A_n. The cases of real growth rate and complex-valued growth rate are treated separately. In either case the disturbance can be expressed directly in terms of the appropriate transform functions. Conclusions are drawn from the form of this function.

Note. Those readers who are interested only in the main results of this analysis can skim the formulation of the problem in §§2 and 3 and then skip directly to §5, which starts with a brief résumé of the analysis.

2. Formulation of the problem and recapitulation of known results

We consider a general class of problems of the form

$$L\mathbf{\Phi} - \frac{\partial}{\partial t} S\mathbf{\Phi} = \sum_{l=1}^{N} (P^{(l)}\mathbf{\Phi} . K^{(l)})\, Q^{(l)}\mathbf{\Phi}, \tag{2.1}$$

where $\mathbf{\Phi}$ is a real-valued vector $(\Phi_1, \Phi_2, \ldots, \Phi_n)$ defined on a domain

$$D = \{(\xi, \eta, t) \mid -\infty < \xi < \infty, \quad 0 \leqslant \eta \leqslant 1, \quad t \geqslant 0\},$$

ξ and η are spatial variables, and t is the time. Here $L, S, P^{(l)}, Q^{(l)}$ are real linear $n \times n$ matrix partial differential operators, and $K^{(l)}$ is a real linear $n \times 1$ matrix partial differential operator. All operators are assumed to be independent of time and invariant with respect to translation in ξ. The $n \times n$ matrix operator $(P^{(l)}\mathbf{\Phi} . K^{(l)})$ is defined as follows. Let $p_{ij}^{(l)}$ be the elements of $P^{(l)}$, and $k_j^{(l)}$ the components of $K^{(l)}$. Then the vector $\theta^{(l)} = P^{(l)}\mathbf{\Phi}$ has components $\theta_i^{(l)} = \sum_{j=1}^{n} p_{ij}^{(l)} \Phi_j$, and $(P^{(l)}\mathbf{\Phi} . K^{(l)})$ is the matrix with components $\theta_i^{(l)} k_j^{(l)}$.

Boundary conditions on $\mathbf{\Phi}$ are imposed at $\eta = 0$ and $\eta = 1$. They are linear, independent of t, and invariant with respect to ξ. It is understood that all variables have been made appropriately dimensionless.

It is assumed that the boundary-value problem consisting of (2.1) and the specified boundary conditions has a solution $\mathbf{\Phi} = \mathbf{\Phi}_0(\eta)$. We term this solution the basic flow. The stability of the basic flow is investigated by superimposing a distance $\mathbf{\Phi}'(\xi, \eta, t)$. Setting $\mathbf{\Phi} = \mathbf{\Phi}_0 + \mathbf{\Phi}'$ in (2.1) we obtain

$$\left(\mathscr{L} - \frac{\partial}{\partial t} S\right) \mathbf{\Phi}' = \sum_{l=1}^{N} (P^{(l)}\mathbf{\Phi}' . K^{(l)})\, Q^{(l)}\mathbf{\Phi}', \tag{2.2}$$

where $\mathscr{L}$ defined by

$$\mathscr{L}\mathbf{\Phi}' = L\mathbf{\Phi}' + \sum_{l=1}^{N} [(P^{(l)}\mathbf{\Phi}_0 . K^{(l)})\, Q^{(l)}\mathbf{\Phi}' + (P^{(l)}\mathbf{\Phi}' . K^{(l)})\, Q^{(l)}\mathbf{\Phi}_0] \tag{2.3}$$

is a real linear $n \times n$ matrix partial differential operator with the same properties as L. Also $\mathscr{L}$ will depend on the parameter R. The boundary conditions for $\mathbf{\Phi}'$ at $\eta = 0$ and $\eta = 1$ are now homogeneous.

The problem consisting of the differential equation (2.2), the homogeneous boundary conditions, and appropriate initial conditions has been discussed in some detail by Eckhaus (1965) for the case of a scalar function; and that work has been generalized to the present problem by Kogelman & DiPrima (1970). These authors have considered (*a*) the linear stability problem; (*b*) the non-linear supercritical equilibrium solution or subcritical instability of a disturbance which is periodic in ξ with period $2\pi/k$; and (*c*) the linear stability theory of supercritical equilibrium states. In order to establish notation and to lay the foundation for succeeding generalizations it is necessary to recapitulate the results of these considerations.

Consider first the linear problem, in which the terms on the right-hand side of (2.2) are neglected. As has been mentioned, there are solutions of the form

$$\mathbf{\Phi}'(\xi, \eta, t) = \epsilon \boldsymbol{\phi}^{(k)}(\eta) \exp[-ik\xi - \mu^{(k)}t], \tag{2.4}$$

where k is a non-negative real number, $\epsilon = \epsilon(k, R)$ is a parameter and in general $\mu^{(k)}$ is complex. Substitution into (2.2) with the terms on the right-hand side neglected gives

$$(\mathscr{L}_k + \mu^{(k)} S_k)\, \boldsymbol{\phi}^{(k)} = 0, \tag{2.5}$$

where the definitions of $\mathscr{L}_k$ and S_k are clear. For a given value of k, (2.5) together with the homogeneous boundary conditions, defines an eigenvalue problem for $\mu^{(k)}$ in which R appears (through $\mathscr{L}_k$) as a parameter. In general the eigenvalue problem will be non-selfadjoint. We shall assume that there exists a denumerable infinity of eigenvalues $\mu_m^{(k)}$, with no cluster point in the finite plane, which can be ordered for each value of k so that

$$\tau_{m+1}^{(k)} \geqslant \tau_m^{(k)}, \quad \text{where} \quad \tau_m^{(k)} = \operatorname{Re} \mu_m^{(k)}. \tag{2.6}$$

The corresponding eigenvectors $\boldsymbol{\phi}_m^{(k)}(\eta)$, some of which may be generalized eigenvectors, are assumed to be complete in a certain Hilbert space H_S and to satisfy the biorthogonality condition

$$(S_k \boldsymbol{\phi}_m^{(k)}, \tilde{\boldsymbol{\phi}}_n^{(k)}) = \begin{cases} 1 & \text{if} \quad n = m, \\ 0 & \text{if} \quad n \neq m, \end{cases} \tag{2.7}$$

with the adjoint eigenvectors $\boldsymbol{\phi}_n^{(k)}$ of the adjoint eigenvalue problem

$$(\tilde{\mathscr{L}}_k + \lambda^{(k)} \tilde{S}_k)\, \tilde{\boldsymbol{\phi}}^{(k)} = 0$$

and the adjoint boundary conditions. Here (,) is the naturally occurring inner product; see, for example, Kogelman & DiPrima (1970).

DiPrima & Habetler (1969) have given sufficient conditions, which are satisfied by each of the specific problems mentioned earlier, for the above properties of the eigenvalues and eigenfunctions to hold. In the class of problems they considered, the operator S_k was positive definite, the Hilbert space H_S was the completion of the pre-Hilbert space with domain equal to that of S_k and inner product $[\boldsymbol{\phi}^{(k)}, \boldsymbol{\psi}^{(k)}] = (\boldsymbol{\phi}^{(k)}, S_k \boldsymbol{\psi}^{(k)})$, if $\boldsymbol{\phi}^{(k)}$ and $\boldsymbol{\psi}^{(k)}$ are in the domain of S_k. Further, $\mathscr{L}_k$ could be written as the sum of an operator which is positive and bounded below and an operator B_k such that $S_k^{-1} B_k$ is bounded in H_s.

As mentioned in §1 we assume that the lowest eigenvalue $\mu_0^{(k)}$ is simple, and that the neutral curve $R = R_c(k)$ defined by $\tau_0^{(k)}(R) = 0$ has a minimum R_c corresponding to k_c. See figure 1 (*a*).

The full solution of the linear stability problem for a disturbance which is periodic with period $2\pi/k$ is

$$\boldsymbol{\Phi}'(\xi, \eta, t) = \sum_{n=-\infty}^{\infty} \boldsymbol{\Phi}_{nk}(\eta, t)\, e^{-ink\xi} \tag{2.8a}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \epsilon A_m^{(nk)} \boldsymbol{\phi}_m^{(nk)}(\eta) \exp -[ink\xi - \mu_m^{(nk)} t]. \tag{2.8b}$$

(Here, as in the remainder of the paper, it is understood that $\boldsymbol{\Phi}_{nk}$ for n negative is the complex conjugate of $\boldsymbol{\Phi}_{nk}$ for n positive.) The constants $\epsilon A_m^{(nk)}$ are determined by the initial values, $\boldsymbol{\Phi}^{(nk)}(\eta, 0)$, which must lie in H_S. Thus the Hilbert space H_S determines the class of admissible disturbances, and the completeness of the eigenvectors allows the expansion (2.8*b*).

Next, let us consider solutions of the non-linear problem which have spatial period $2\pi/k$. We assume that R is near $R_c(k)$ so that the parameter

$$\epsilon^2(k, R) = |\tau_0^{(k)}(R)|,$$

which we introduced in (1.5), is small. For definiteness it is helpful to fix attention on the slightly supercritical case where R and k are such that (for positive integers n and p) $\tau_p^{(nk)}(R) > 0$ except when $p = 0, n = 1$; $\tau_0^{(k)}(R) < 0$. In this case, according to linear theory only the fundamental grows in time while all its harmonics are predicted to decay. The analysis is also valid in the slightly subcritical case when $R < R_c(k)$ and ϵ^2 is small, where all modes decay according to linear theory, but the fundamental decays very slowly.

To construct a formal asymptotic solution of the governing non-linear equations for a disturbance that is periodic in ξ with period $2\pi/k$, by an expansion in terms of ϵ, we substitute the series (2.8) for $\boldsymbol{\Phi}'$ into (2.2). In the first step of the analysis an infinite set of coupled non-linear partial differential equations for the $\boldsymbol{\Phi}_{nk}$ is obtained. The second step is to scale the $\boldsymbol{\Phi}_{nk}$ to reflect the fact that if $\tau_0^{(k)} < 0$ then the fundamental $\boldsymbol{\Phi}_k$ can grow freely according to linear theory, while all of the harmonics $\boldsymbol{\Phi}_{nk}$, $n \neq 1$, continue to exist only in so far as they are forced by the growth of the fundamental. It turns out that the appropriate scaling is

$$\boldsymbol{\Phi}_{nk} = \epsilon^{1+|n-1|} \boldsymbol{\Psi}_{nk}; \quad \boldsymbol{\Psi}_{nk} = O(1). \tag{2.9}$$

With this scaling it can be shown that to $O(\epsilon^2)$ the equations for $\boldsymbol{\Psi}_0$ and $\boldsymbol{\Psi}_{2k}$ reduce to linear partial differential equations whose respective non-homogeneous terms are quadratic in $\boldsymbol{\Psi}_k$. Further, $\boldsymbol{\Psi}_k$ satisfies an equation of the form

$$\left(\mathscr{L}_k + \frac{\partial}{\partial t} S_k\right) \boldsymbol{\Psi}_k = \epsilon^2 \mathbf{F}, \tag{2.10}$$

where $\mathbf{F}$ is quadratic in $\mathbf{\Psi}_0$, $\mathbf{\Psi}_k$, and $\mathbf{\Psi}_{2k}$, $\mathbf{\Psi}_k$. Thus to $O(\epsilon^2)$ the equations for $\mathbf{\Psi}_0(\eta,t)$, $\mathbf{\Psi}_k(\eta,t)$ and $\mathbf{\Psi}_{2k}(\eta,t)$ are self-consistent and allow $\mathbf{\Phi}'$ to be computed up to $O(\epsilon^3)$ (since $\mathbf{\Phi}_{nk} = O(\epsilon^2)$ if $n > 2$). Substituting

$$\mathbf{\Psi}_k(\eta,t) = \sum_{m=0}^{\infty} A_m^{(k)}(t)\,\boldsymbol{\phi}_m^{(k)}(\eta)$$

in (2.10), taking the inner product with $\tilde{\boldsymbol{\phi}}_m^{(k)}$, and recalling that $\tau_m^{(k)} = O(1)$ if $m \neq 0$, we find

$$\mathbf{\Psi}_k(\eta,t) = A_0^{(k)}(t)\,\boldsymbol{\phi}_0^{(k)}(\eta) + O(\epsilon^2). \tag{2.11}$$

It then follows from the equations for $\mathbf{\Psi}_0$ and $\mathbf{\Psi}_{2k}$ that

$$\mathbf{\Psi}_0(\eta,t) = |A_0^{(k)}(t)|^2\,\mathbf{G}_0(\eta) + O(\epsilon^2), \quad \mathbf{\Psi}_{2k}(\eta,t) = |A_0^{(k)}(t)|^2\,\mathbf{G}_2(\eta) + O(\epsilon^2), \tag{2.12}$$

where $\mathbf{G}_0(\eta)$ and $\mathbf{G}_2(\eta)$ are determinate solutions of ordinary differential equations. Finally the amplitude $A_0^{(k)}(t)$ satisfies the Landau equation

$$dA_0^{(k)}/dt + \mu_0^{(k)} A_0^{(k)} = -\epsilon^2 \beta^{(k)} A_0^{(k)} |A_0^{(k)}|^2 + O(\epsilon^4), \tag{2.13}$$

where $\beta^{(k)}$, the Landau constant, is a determinate parameter. See Eckaus (1965) and Kogelman & DiPrima (1970) for specific details and formulas. In particular, these authors show that *provided* $\beta^{(k)}$ *is of order unity*, the terms neglected in (2.13) are of higher order than those which are retained. From (2.8) and (2.9), when the solution of (2.13) is inserted into (2.11) and (2.12), we thus obtain a solution for the perturbation $\mathbf{\Phi}'$ which is correct through terms of order ϵ^2.

Equation (2.13) possesses a periodic solution of the form $A_0^{(k)} = A\exp(i\omega t)$, where

$$A^2 = [\mathrm{Re}\,\beta^{(k)}]^{-1}\,\mathrm{sgn}\,\tau_0^{(k)} + O(\epsilon^2), \quad \omega = -\mathrm{Im}\,(\mu_0^{(k)}) + \epsilon^2 A^2\,\mathrm{Im}\,(\beta^{(k)}) + O(\epsilon^4). \tag{2.14}$$

If $\mathrm{Re}\,[\beta^{(k)}] < 0$, then in order for A^2 to be positive we must have $\tau_0^{(k)} < 0$ which requires $R > R_c$. In this case an analysis in the phase plane shows that

$$\lim_{t\to\infty} |A_0^{(k)}(t)|^2 = A^2.$$

Thus a disturbance which initially grows exponentially according to linear theory tends to the following supercritical equilibrium solution as $t \to \infty$:

$$\mathbf{\Phi}'(\xi,\eta,t) = 2\epsilon A\,\mathrm{Re}\,[e^{-i(k\xi-\omega t)}\boldsymbol{\phi}_0^{(k)}(\eta)] + \epsilon^2 A^2[\mathbf{G}_0(\eta) + 2\,\mathrm{Re}\,e^{-i2(k\xi-\omega t)}\mathbf{G}_2(\eta)] + O(\epsilon^3). \tag{2.15}$$

If $\mathrm{Re}\,(\beta^{(k)}) > 0$, then although the solution (2.14) exists when $R < R_c$, we have $\lim_{t\to\infty} |A_0^{(k)}(t)| = 0$ if the initial value of $|A_0^{(k)}(t)|^2$ is less than A^2, and $\lim_{t\to\infty} |A_0^{(k)}(t)| = \infty$ if the initial value of $|A_0^{(k)}(t)|^2$ is greater than A^2. Thus while for $R < R_c$ the basic flow is stable according to linear theory it is actually unstable to disturbances of sufficient magnitude. The basic flow is said to be subject to a subcritical instability.

The stability of the supercritical equilibrium flow (2.15) to $O(\epsilon^3)$ noise composed of arbitrary wave-numbers k (as depicted schematically in figure 2) has also been investigated by Eckhaus (1965) with generalizations by Kogelman &

DiPrima (1970). The restriction that the noise is $O(\epsilon^3)$ allows a linearization of the stability problem. As mentioned in §1, for $R-R_c$ small and positive super-critical equilibrium flows of the form (2.15) are stable provided that

$$(k_1(R)-k_c)/\sqrt{3} < k-k_c < (k_2(R)-k_c)/\sqrt{3}. \tag{2.16}$$

See figure 1 (*b*). A similar result was recently obtained in a statistical framework by Newell, Lange & Aucoin (1970).

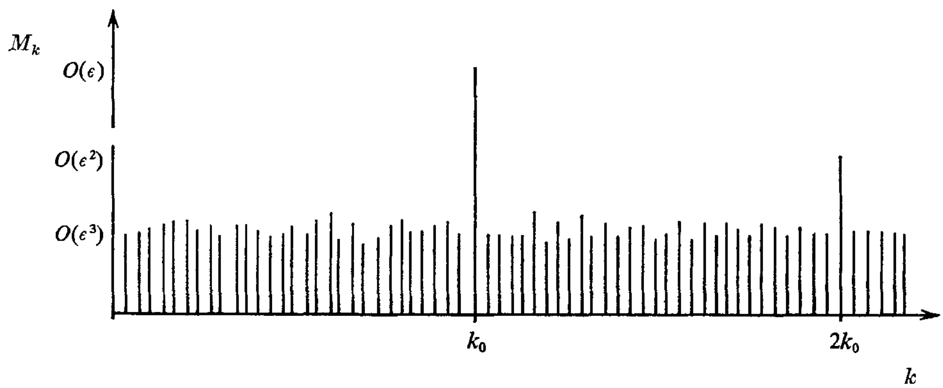

FIGURE 2. Schematic depiction of the initial amplitudes in the stability analysis of a super-critical equilibrium flow with spatial period $2\pi/k_0$, as in Eckhaus (1965). M_k is the initial amplitude of the Fourier component with wave-number k. The scale on the ordinate is schematically indicated in orders of magnitude. The small parameter ϵ^2 is proportional to $R-R_c(k_0)$.

3. Non-linear interactions

We wish to investigate the non-linear mechanism for wave-number selection in a competition among a large number of modes having different wave-numbers but with *comparable initial magnitudes*. The Fourier representation of these modes and all of their interactions can be written as

$$\mathbf{\Phi}'(\xi,\eta,t) = \sum_{m=-\infty}^{\infty} \mathbf{\Phi}_{m\Delta k}(\eta,t)\, e^{-im\Delta k\xi}, \tag{3.1}$$

where
$$\mathbf{\Phi}_{m\Delta k}(\eta,t) = \frac{\Delta k}{2\pi}\int_{-\pi/\Delta k}^{\pi/\Delta k} e^{im\Delta k\xi}\,\mathbf{\Phi}'(\xi,\eta,t)\,d\xi, \tag{3.2}$$

and Δk is small in a sense to be made precise later. Notice that a disturbance $\mathbf{\Phi}'$ of period $2\pi/k$ in ξ can be represented by (3.1) by taking $\mathbf{\Phi}_{m\Delta k}(\eta,t) = 0$ if $m \neq nj$ and $\mathbf{\Phi}_{m\Delta k}(\eta,t) \neq 0$ if $m = nj$, for all integers n and with $j\Delta k = k$.

Substituting (3.1) in (2.2), multiplying by $(2\pi)^{-1}\Delta k \exp(in\xi\,\Delta k)$, and integrating from $-\pi/\Delta k$ to $\pi/\Delta k$, we find the following set of non-linear partial differential equations (for each component $\mathbf{\Phi}_k$):

$$\left(\mathscr{L}_k - \frac{\partial}{\partial t}S_k\right)\mathbf{\Phi}_k = \sum_{l=1}^{N}\mathbf{F}_k^{(l)}, \tag{3.3}$$

where $k = m\Delta k$, $m = 0, \pm 1, \pm 2, \ldots$. Here

$$\mathbf{F}_0^{(l)} = (P_0^{(l)}\mathbf{\Phi}_0 . K_0^{(l)})\, Q_0^{(l)}\mathbf{\Phi}_0 + \sum_{k'=\Delta k}^{\infty} \{(\bar{P}_{k'}^{(l)}\bar{\mathbf{\Phi}}_{k'} . K_{k'}^{(l)})\, Q_{k'}^{(l)}\mathbf{\Phi}_{k'} + (P_{k'}^{(l)}\mathbf{\Phi}_{k'} . \bar{K}_{k'}^{(l)})\, \bar{Q}_{k'}\bar{\mathbf{\Phi}}_{k'}\}, \quad (3.4a)$$

and for $k \neq 0$,

$$F_k^{(l)} = (P_0^{(l)}\mathbf{\Phi}_0 . K_k^{(l)})\, Q_k^{(l)}\mathbf{\Phi}_k + (P_k^{(l)}\mathbf{\Phi}_k . K_0^{(l)})\, Q_0^{(l)}\mathbf{\Phi}_0 + \sum_{k'=\Delta k}^{k-\Delta k} [(P_{k'}^{(l)}\mathbf{\Phi}_{k'} . K_{k-k'}^{(l)})\, Q_{k-k'}^{(l)}\mathbf{\Phi}_{k-k'}] + \sum_{k'=\Delta k}^{\infty} [(\bar{P}_{k'}^{(l)}\bar{\mathbf{\Phi}}_{k'} . K_{k+k'}^{(l)})\, Q_{k+k'}^{(l)}\mathbf{\Phi}_{k+k'} + (P_{k+k'}^{(l)}\mathbf{\Phi}_{k+k'} . \bar{K}_{k'})\, \bar{Q}_{k'}\bar{\mathbf{\Phi}}_{k'}]. \quad (3.4b)$$

The definitions of $\mathscr{L}_k$, S_k, $P_k^{(l)}$, $Q_k^{(l)}$ and $P_k^{(l)}\mathbf{\Phi}_k . K_k^{(l)}$ should be clear, and a bar denotes the complex conjugate. The sums are taken over all consecutive values of k' (a distance Δk apart) between the indicated limits.

Before going on with the main analysis we must consider several preliminary points. The first of these is a discussion of the limiting procedure to be employed, the associated small parameters, and a study of the neutral curve near (k_c, R_c). In addition we introduce the concept of free modes.

Modes with wave-numbers near k_c are the most rapidly growing according to linear theory. Let k_0 be any wave-number within $O[\epsilon^2(k_0, R)]$ of k_c and choose Δk so that k_0 is an integer multiple of Δk. In a sense which will soon be clear, we are now going to 'centre' our analysis on the mode with wave-number k_0. Thus for the remainder of the paper we write ϵ for $\epsilon(k_0, R) = |\tau_0^{(k_0)}(R)|^{\frac{1}{2}}$. The asymptotic analysis as $\epsilon \to 0$ to be presented is easiest to envision for the special case $k_0 = k_c$ wherein the asymptotic expansion is explicitly with respect to the limit $R \to R_c$. But as $\epsilon \to 0$ the requirement $k_c - k_0 = O(\epsilon^2)$ means that $k_0 \to k_c$ and hence $R \to R_c$ (and $\epsilon_c \to 0$). (This requirement will also be needed in §5.) The analysis is meaningful for small ϵ, and added generality is obtained by centring on k_0 rather than k_c.

We consider the small quantity $\tau_0^{(k)}(R)$ as a function of k and R in more detail. Recall that the equation $\tau_0^{(k)}(R) = 0$ implicitly defines the neutral curve $R = R_c(k)$. It follows that the derivative of $\tau_0^{(k)}(R)$ along the neutral curve is zero, and in particular

$$[\partial \tau_0^{(k)}(R)/\partial k]_{k=k_c,\, R=R_c} = 0.$$

If we expand $\tau_0^{(k)}(R)$ about k_0 we find

$$\tau_0^{(k)}(R) = \tau_0^{(k_0)}(R) + \left.\frac{\partial \tau_0^{(k)}(R)}{\partial k}\right|_{k=k_0} (k-k_0) + \frac{1}{2}\left.\frac{\partial^2 \tau_0^{(k)}(R)}{\partial k^2}\right|_{k=k_0} (k-k_0)^2 + \ldots. \quad (3.5a)$$

Next expanding $[\partial \tau_0^{(k)}(R)/\partial k]_{k=k_0}$ about (k_c, R_c) and making use of the above equation gives

$$\begin{aligned}\tau_0^{(k)}(R) &= \tau_0^{(k_0)}(R) + [O(k_0 - k_c) + O(R - R_c)]\,(k-k_0) + O(k-k_0)^2 \\ &= \tau_0^{(k_0)}(R) + O[\epsilon^2(k-k_0)] + O[\epsilon_c^2(k-k_0)] + O(k-k_0)^2. \quad (3.5b)\end{aligned}$$

This equation will be used in §5. Of immediate concern is the fact that if $k = k_c$ then

$$\tau_0^{(k_c)}(R) = \tau_0^{(k_0)}(R) + O(\epsilon^4) + O(\epsilon^2\epsilon_c^2),$$

which shows that ϵ_c and ϵ are the same up to an error $O(\epsilon)$ as $\epsilon \to 0$.

For R slightly greater than R_c any mode with wave-number k in (k_1, k_2) will grow according to linear theory; see figure 1 (*b*). The width of this interval can be expressed as follows. In the neighbourhood of (k_c, R_c) the neutral curve can be approximated by a parabola so that if $R - R_c = O(\epsilon_c^2)$ then $k_2(R) - k_1(R) = O(\epsilon_c)$. This interval can also be thought of as an interval of width $O(\epsilon)$ centred on k_0. We are interested in studying the interaction of a number of modes with wave-numbers in this interval. Thus we require $\Delta k = O(\epsilon)$ which does not preclude the possibility that $\Delta k = o(\epsilon)$.

We denote by J the set of wave-numbers which are within $O(\epsilon)$ of k_0. The corresponding components $\mathbf{\Phi}_k(\eta, t)$ in the Fourier series (3.1) are called the *most dangerous modes*. They comprise (*a*) all modes which grow, and (*b*) the modes of slowest decay – according to linear theory. For such modes $\tau_0^{(k)}(R) = O(\epsilon^2)$ from (3.5*b*).

In the analyses to follow we shall make extensive use of *free modes*, denoted by $\epsilon^3 \boldsymbol{\chi}_k$. These functions are required to satisfy the linearized equations for components of wave-number k as well as certain $O(\epsilon^3)$ initial conditions. Thus

$$\boldsymbol{\chi}_k(\eta, t) = \sum_{n=0}^{\infty} C_n^{(k)} \boldsymbol{\phi}_n^{(k)}(\eta) \exp\left[-\mu^{(k)} t\right],$$

where the eigenvectors $\boldsymbol{\phi}_n^{(k)}$ satisfy (2.5) and the $C_n^{(k)}$ are $O(1)$ constants to be determined by the initial conditions. We shall only use free modes for $k \notin J$. For $k \notin J$ the free modes decay exponentially with time. We note the following important facts (Eckhaus 1965, chapter 8): if we write $\mathbf{\Phi}_k = \epsilon^3 \boldsymbol{\chi}_k + \mathbf{\Phi}_k^{(F)}$ then the $\mathbf{\Phi}_k^{(F)}$ satisfy non-linear equations which contain forced terms due to non-linear interactions, but the free modes do not contribute to the forcing terms at lowest order.

With the aid of the foregoing preliminaries, we can now formulate the central problem of this paper. We shall be concerned with a disturbance whose dominant terms initially have magnitudes which are $O(\epsilon)$ and are a linear combination of the most dangerous modes. That is, $\mathbf{\Phi}_k(\eta, 0) = O(\epsilon)$ if $k \in J$. Our first step is to select scales for the various Fourier components which represent the largest magnitude which a given component is expected to attain during the course of time, given that the most dangerous modes are $O(\epsilon)$.

To modes corresponding to wave-numbers not near 0, k_0, $2k_0$ we assign initial conditions $O(\epsilon^3)$. Such initial conditions can be satisfied with free modes which, as we have already noted, need not be considered in determining the leading effect of the non-linear interactions. Furthermore, as we shall see, non-linear interactions never cause the modes in question to grow to a magnitude larger than that assigned to them initially. Thus these modes will be assigned a scale $O(\epsilon^3)$, and this scale is expected to be valid uniformly in time.

We turn to the scaling of the remaining modes. Consider the particular case of the mean motion term ($k = 0$). Suppose this mode was initially $O(\epsilon^2)$. In spite of the decaying effect of initial conditions, it would later reach a larger $O(\epsilon^2)$ magnitude due to the interaction of $O(\epsilon)$ modes near k_0. Following this type of reasoning, we assign $O(\epsilon^2)$ initial conditions to modes with wave-numbers near zero and near $2k_0$. We choose an $O(\epsilon^2)$ scale for these modes, and we expect this scale to be valid uniformly in time.

Next it is necessary to consider transition regions linking $O(\epsilon)$ modes with wave-numbers near k_0 with $O(\epsilon^3)$ modes with wave-numbers somewhat farther from k_0, and linking $O(\epsilon^2)$ modes with wave-numbers near zero and $2k_0$ with $O(\epsilon^3)$ modes with wave-numbers somewhat farther away from zero and $2k_0$.

Consider a transition region called J^* where $\epsilon < O(k-k_0) < 1$.† As $k-k_0$ increases from slightly greater than $O(\epsilon)$ to $O(1)$ the initial conditions for the corresponding $\mathbf{\Phi}_k$ will be required to decrease in magnitude from $o(1)$ to $O(\epsilon^3)$. Such initial conditions can be regarded as the sum of two terms, one $O(\epsilon^3)$ and one $O[\epsilon\delta_k(\epsilon)]$, providing that (as a minimum requirement) the magnitude of the $\delta_k(\epsilon)$ decreases from $o(1)$ to $O(\epsilon^3)$ as $k-k_0$ increases from slightly greater than $O(\epsilon)$ to $O(1)$. We shall employ free modes to satisfy the $O(\epsilon^3)$ part of the initial conditions, writing

$$\mathbf{\Phi}_k = \epsilon^3\boldsymbol{\chi}_k + \epsilon\delta_k(\epsilon)\mathbf{\Psi}_k \quad (k \in J^*).$$

We shall find it necessary to require much more rapid decay of the $\delta_k(\epsilon)$ than that given by the above-stated minimum requirement.

With this in mind we define the following sets of wave-numbers and corresponding scaled Fourier components $\mathbf{\Psi}_k(\eta, t)$.

$$\begin{aligned}
J &= \{k \mid k = k_0 + O(\epsilon)\}: \quad \mathbf{\Phi}_k = \epsilon\delta_k(\epsilon)\mathbf{\Phi}_k,\ \delta_k = 1.\\
J^* &= \{k \mid \epsilon < O(k-k_0) < 1\}: \quad \mathbf{\Phi}_k = \epsilon^3\boldsymbol{\chi}_k + \epsilon\delta_k(\epsilon)\mathbf{\Psi}_k,\ \delta_k = o(1).\\
I &= \{k \mid k = O(\epsilon)\}: \quad \mathbf{\Phi}_k = \epsilon^2\delta_k(\epsilon)\mathbf{\Psi}_k,\ \delta_k = 1.\\
I^* &= \{k \mid \epsilon < O(k) < 1\}: \quad \mathbf{\Phi}_k = \epsilon^3\boldsymbol{\chi}_k + \epsilon^2\delta_k(\epsilon)\mathbf{\Psi}_k,\ \delta_k = o(1).\\
Y &= \{k \mid k = 2k_0 + O(\epsilon)\}: \quad \mathbf{\Phi}_k = \epsilon^2\delta_k(\epsilon)\mathbf{\Psi}_k,\ \delta_k = 1.\\
Y^* &= \{k \mid \epsilon < O(k-2k_0) < 1\}: \quad \mathbf{\Phi}_k = \epsilon^3\boldsymbol{\chi}_k + \epsilon^2\delta_k(\epsilon)\mathbf{\Psi}_k,\ \delta_k = o(1).\\
&\text{For all other } k: \quad \mathbf{\Phi}_k = \epsilon^3\boldsymbol{\chi}_k + \epsilon^3\mathbf{\Psi}_k. \qquad (3.6)
\end{aligned}$$

Although $\delta_k(\epsilon) = 1$ for $k \in J,\ I,\ Y$, retention of these δ_k simplifies some later equations. As indicated in (3.6), $\delta_k(\epsilon) = o(1)$ for $k \in J^*,\ I^*,\ Y^*$.

Motivated by our earlier discussion, we take the initial values of the $\mathbf{\Psi}_k$ to be zero for $k \notin J, J^*, I, I^*, Y$ and Y^*, for the initial conditions can be satisfied with the $\boldsymbol{\chi}_k$. Elsewhere, the $\mathbf{\Psi}_k$ may be $O(1)$ initially. All the $\boldsymbol{\chi}_k$ may be $O(1)$ initially. We shall thus consider an initial-value problem wherein the modes have magnitudes as depicted in figure 3.

It is the central task of this paper, then, to trace the development of the disturbance whose modal amplitudes have magnitudes as given in figure 3. By our choice of scales, we anticipate that no mode will ever have a magnitude exceeding its initial one. Thus we assert that figure 3 can also serve to depict an order of magnitude estimate which is uniformly valid in time.

Validating this assertion requires several steps. In §4 we shall demonstrate by a careful order of magnitude analysis that $\mathbf{\Psi}_k(\eta, t) = O(1)$ for $k \notin J,\ I,\ Y$ in any time interval in which $\mathbf{\Psi}_k(\eta, t) = O(1)$ for $k \in J,\ I,\ Y$. In the present section we use this result to analyze $\mathbf{\Psi}_k$ when $k \in J, I,\ Y$. We show that $\mathbf{\Psi}_k(\eta, t) = O(1)$

† The notation $\epsilon < O(k-k_0) < 1$ means that $k-k_0 = o(1)$ and $\epsilon = o(k-k_0)$ as $\epsilon \to 0$.

for $k \in I$, Y in any time interval in which $\Psi_k(\eta, t) = O(1)$ for $k \in J$. Furthermore, we develop the equations which define approximations to Ψ_k for $k \in J$, I, Y valid under these conditions. Finally, in § 5 we shall show that solutions of the approximate equations indeed define $\Psi_k(\eta, t) = O(1)$ uniformly in time for $k \in J$. This will complete the chain of deductions, showing that the analysis is consistent.

We now face a fairly substantial algebraic calculation. The details of this calculation are recorded in appendices A and B. Here we will simply outline the steps that must be taken. First we substitute the expressions given in (3.6)

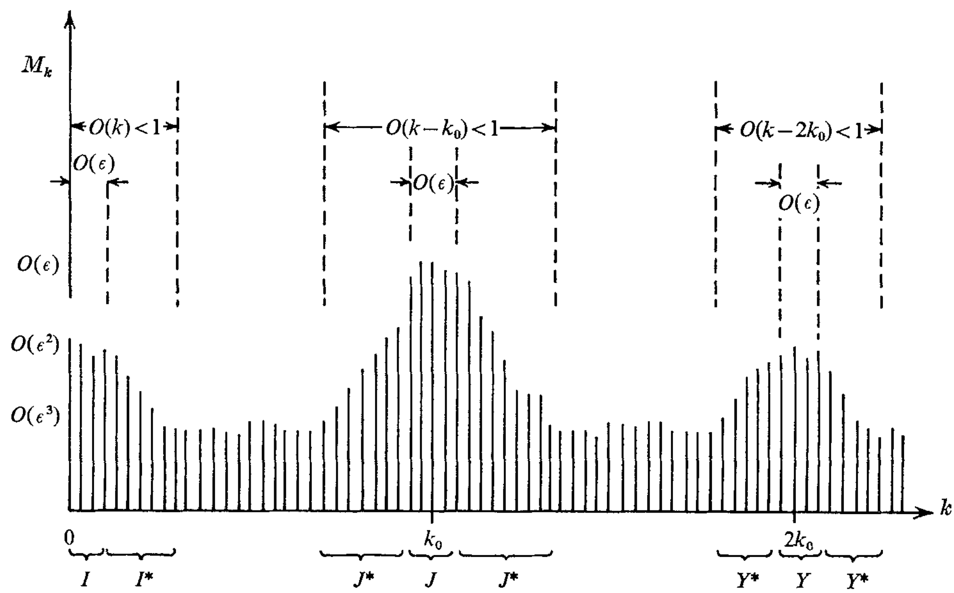

FIGURE 3. Schematic depiction of the initial amplitudes in the present analysis. M_k is the magnitude of the Fourier component with wave-number k. The central wave-number k_0 is within $O(\epsilon^2)$ of k_c. The widths of the wave-number sets J^*, I^* and Y^* are foreshortened. The $O(1)$ widths of the region between J^* and I^* and the region between I^* and Y^* are foreshortened even more.

for Φ_k in (3.3) and obtain to the appropriate order of accuracy the equations for $k = 0$, for $k \in I$ but $k \neq 0$, for $k \in Y$, and for $k \in J$. An examination of these equations shows that, just as for the case of the periodic solution discussed in § 2, once the family of Fourier components Ψ_k for $k \in J$ is known then the non-homogeneous terms in the equations for other k's are known and we can solve for Ψ_0, Ψ_k for $k \in I$ but $k \neq 0$, and Ψ_k for $k \in Y$.

Thus we focus our attention on the equations for $k \in J$, and write

$$\Psi_k(\eta, t) = \sum_{p=0}^{\infty} A_p^{(k)}(t)\, \phi_p^{(k)}(\eta) \quad (k \in J). \tag{3.7}$$

As in § 2 we find that for $k \in J$

$$\Psi_k(\eta, t) = A_0^{(k)}(t)\, \phi_0^{(k)}(\eta) + O(\epsilon^2), \tag{3.8}$$

where
$$\left.\begin{aligned} dA_0^{(k)}/dt + \mu_0^{(k)} A_0^{(k)} &= -\epsilon^2 b_0^{(k)} + O(\epsilon^4), \\ \text{or} \quad d|A_0^{(k)}|^2/dt + 2\tau_0^{(k)} |A_0^{(k)}|^2 &= -\epsilon^2 [A_0^{(k)} \bar{b}_0^{(k)} + \bar{A}_0^{(k)} b_0^{(k)}] + O(\epsilon^4). \end{aligned}\right\} \tag{3.9}$$

Here $b_0^{(k)}(t) = (B_k, \tilde{\phi}_0^{(k)})$ where B_k is the term $O(\epsilon^2)$ on the right-hand side of the equation for $\mathbf{\Psi}_k$ and is composed of terms quadratic in $\mathbf{\Psi}_{k'}$ and $\mathbf{\Psi}_{k''}$ with $k'' \in J$ and with $k' = 0$, or $k' \in I$ but $k' \neq 0$, or $k' \in Y$. See (A4).

Next (3.7) and (3.8) are used to evaluate the non-homogeneous terms (which involve quadratic interactions of the $\mathbf{\Psi}_k$, $k \in J$) in the equations for $\mathbf{\Psi}_k$ with $k = 0$, $k \in I$ but $k \neq 0$, and $k \in Y$. It turns out that $\mathbf{\Psi}_0$, $\mathbf{\Psi}_k$ for $k \in I$ but $k \neq 0$, and $\mathbf{\Psi}_k$ for $k \in Y$ can be expressed in terms of sums of quadratic products of $A_0^{(k)}(t)$ for $k \in J$ multiplied by determinate functions of η. It follows that the B_k for $k \in J$ have similar forms and that finally $b_0^{(k)}(t)$ can be expressed in terms of quadratic products of the $A_0^{(k)}(t)$. The expression for $b_0^{(k)}(t)$ is given in (A 24) of appendix A. This expression is very complicated. However, since $b_0^{(k)}(t)$ appears multiplied by ϵ^2 in (3.9) it is only necessary to evaluate it in the limit as $\epsilon \to 0$; that is, terms $O(\epsilon)$ can be neglected. This results in considerable simplification. The leading approximation for $b_0^{(k)}(t)$ is given in (A 30).

It is convenient now to introduce a notational convenience. The modes in J are separated by Δk and are spread over an interval of width $O(\epsilon)$ centred on k_0 on the K axis. We take $\Delta k = \sigma\epsilon$ where $\sigma = O(1)$, so that the denumerable set of modes in J can be characterized by $k_0 + n\sigma\epsilon$, $n = 0, \pm 1, \pm 2, \ldots$. (The Fourier components in I are at $0, \sigma\epsilon, 2\sigma\epsilon, \ldots$ and those in Y are at $2k_0 + n\sigma\epsilon$, $n = 0, \pm 1, \pm 2, \ldots$.) The parameter σ will be discussed further in § 5; for the present we point out that we do not exclude the possibility that $\sigma = o(1)$ as $\epsilon \to 0$. We write

$$A_0^{(k)} = A_0^{(k_0+n\sigma\epsilon)} = A_n, \tag{3.10a}$$

$$\mu_0^{(k)} = \mu_0^{(k_0+n\sigma\epsilon)} = \mu_0^{(n)}, \tag{3.10b}$$

for $k \in J$. Then making use of the results in the appendices we find that the fundamental system of amplitude equations for the most dangerous modes $(k \in J)$ is

$$dA_n/dt + \mu_0^{(n)} A_n = -\epsilon^2 \beta_c \sum_{p\in J} \sum_{m\in J} A_p \bar{A}_m A_{n+m-p} + O(\epsilon^3) \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{3.11}$$

Here and in the remainder of the paper the summation notation $\sum_{m\in J}$ means a sum over all the wave-numbers $k_0 + m\sigma\epsilon$ in J, and β_c is the Landau constant evaluated at $k = k_c$, $R = R_c$. We see that the error in (3.11) relative to terms retained is $O(\epsilon)$. This relative error would be $O(\epsilon^2)$ if we had included the term $O(\epsilon)$ in the evaluation of $b_0^{(k)}$. We shall defer our major discussion of (3.11) until § 5. None the less, at this stage it is worth emphasizing the significance of the terms in (3.11).

We are concerned with the interactions of a denumerable set of fundamental modes with wave-numbers $k = k_0 + n\sigma\epsilon$, $n = 0, \pm 1, \pm 2, \ldots$, where $k_0 = k_c + O(\epsilon^2)$. Each of these modes will in general be assigned a different initial value, but all such initial values must be $O(\epsilon)$ in magnitude. Each mode generates its own first harmonic and contribution to the mean. These have magnitude $O(\epsilon^2)$. There are interactions among the fundamentals which generate $O(\epsilon^2)$ terms as well as interactions which generate higher-order terms, as illustrated in figure 3. Modes $\mathbf{\Phi}_0^{(k)}$ when $k = O(\epsilon)$ $[k \in I]$ and when $k = 2k_0 + O(\epsilon)$ $[k \in Y]$ interact with the fundamental set $[k \in J]$; the influence of these interactions is felt through the Landau constant β_c. All other modes do not influence the controlling fundamental modes whose amplitudes are given by (3.11).

In a Fourier analysis of the motion the Fourier components associated with wave-numbers $k_0 + n\sigma\epsilon \in J$ are given by

$$\begin{aligned}\mathbf{\Phi}_{k_0+n\sigma\epsilon}(\eta, t) &= \epsilon \mathbf{\Psi}_{k_0+n\sigma\epsilon}(\eta, t) \\ &= \epsilon A_n(t)\, \boldsymbol{\phi}_0^{(k_0+n\sigma\epsilon)}(\eta) + O(\epsilon^2),\end{aligned} \tag{3.12}$$

and the A_n satisfy (3.11). The error in (3.12) is $O(\epsilon^2)$ since the error in the A_n is $O(\epsilon)$. Once the fundamental amplitudes $A_n(t)$ are determined, all first interactions associated with the fundamental family of disturbances can be computed to first order. (See appendix A.)

If the non-linear terms in (3.11) are neglected, we obtain the standard result of linear theory:

$$dA_n/dt + \mu_0^{(n)} A_n = 0, \quad A_n(t) = A_n(0) \exp(-\mu_0^{(n)} t).$$

Let us now consider the special case of (3.11) wherein all the A_n except one, say A_q, are identically zero. We then obtain

$$dA_q/dt + \mu_0^{(q)} A_q = -\epsilon^2 \beta_c A_q |A_q|^2.$$

Remembering the notation introduced in (3.10*a*) we see that the above equation is almost identical with (2.13), the equation governing the amplitude of a periodic disturbance of wave-number k. By making the proper identification $k = k_0 + q\sigma\epsilon$ the equations become identical except for the fact that $\beta^{(k)}$ appears in (2.13) while β_c appears above. But since $k - k_0 = O(\epsilon)$ and assuming that $\beta_c = O(1)$ it is consistent with the neglect of higher-order terms in (2.13) to approximate $\beta^{(k)}$ by $\beta^{(k_c)} = \beta_c$. We thus see again that β_c is identical with the Landau constant which is determined by a solvability condition in the 'classical' non-linear stability analysis of a disturbance of spatial period $2\pi/k_c$.

To sum up, two parameters appear in (3.11). One of them, $-\mu_0^{(n)}$, is the growth rate of a perturbation of wave-number $k_0 + n\sigma\epsilon$. (This parameter depends on R. Of course, $|\mathrm{Re}\,(\mu_0^{(0)})| = \epsilon^2$ is of special importance.) The other, β_c, is a Landau constant. To the order of accuracy given here this parameter is independent of R and is independent of wave-number; it can be determined once and for all, given a physical problem which fits into the framework of our analysis.

We now turn to the postponed matter of the uniform validity of the scaling given by (3.6).

4. Scaling

For reasons stated in § 1 it is not at all clear that energy which is initially largely confined to the most dangerous modes will not spread to the entire spectrum of modes as time increases. The heart of our analysis, therefore, is a demonstration that when the real part of the Landau constant β_c is positive and $O(1)$ then it is consistent to regard as uniformly valid in time the initial order of magnitude estimates given by (3.6). In this section we sketch the rather complicated and detailed examination of the governing partial differential equations which shows that consistency follows from the boundedness of solutions of the system (3.11). This boundedness is demonstrated in § 5.

It is convenient to write

$$\mathbf{\Phi}_k(\eta,t) = \epsilon^3 \boldsymbol{\chi}_k(\eta,t) + \epsilon d_k(\epsilon)\,\boldsymbol{\psi}_k(\eta,t) \quad (k \notin J). \tag{4.1}$$

The $\boldsymbol{\chi}_k$ are the $O(1)$ exponentially decaying 'free' modes which were discussed above (3.6). The quantities d_k and $\boldsymbol{\psi}_k$ are closely related to the quantities δ_k and $\mathbf{\Psi}_k$ defined in the scaling (3.6). The reader is advised not to be concerned about the (easily obtained) exact relationship; he should accept that there are slight advantages in starting the present discussion with the new scaling (4.1).

When $k \in J$, the distinction between free and forced contributions is no longer of value. For $k \in J$ we shall assume that the $\mathbf{\Phi}_k$ are $O(\epsilon)$ uniformly for $0 \leqslant t < \infty$, writing

$$\mathbf{\Phi}_k = \epsilon \boldsymbol{\psi}_k, \quad \boldsymbol{\psi}_k = O(1) \quad \text{for} \quad k \in J. \tag{4.2}$$

(These are the most dangerous modes.)

The goal of this section is to choose the scales $d_k(\epsilon)$ so that the forced contributions $\boldsymbol{\psi}_k$ are $O(1)$ uniformly in time, $k \notin J$, assuming that $\boldsymbol{\psi}_k = O(1)$ for $k \in J$. As was stated in § 3, in doing this we can neglect the free contributions (Eckhaus 1965, chapter 8); the magnitude permitted for the free contributions has been selected specifically to make this neglect possible.

We shall use the notation $d_k(\epsilon) \prec f(\epsilon)$ to signify

$$|d_k(\epsilon)| < Mf(\epsilon) \quad \text{when} \quad 0 < \epsilon < \epsilon_0;$$

while $d_k(\epsilon) \approx f(\epsilon)$ will imply

$$mf(\epsilon) < |d_k(\epsilon)| < Mf(\epsilon) \quad \text{when} \quad 0 < \epsilon < \epsilon_0,$$

for some positive constants m, M and ϵ_0. As in § 3, we also use the notation

$$h_1(\epsilon) < O[f(\epsilon)] < h_2(\epsilon)$$

to mean that as $\epsilon \to 0$, $h_1 = o(f)$, $f = o(h_2)$.

We look for a solution wherein $d_k \prec 1$, so that no modes are initially larger in magnitude than the most dangerous modes.

In determining the behaviour of the scaled (forced) Fourier components $\boldsymbol{\psi}_k$ we expand these functions in series, using the normalized eigenfunctions $\boldsymbol{\phi}_p^{(k)}$:

$$\boldsymbol{\psi}_k(\eta,t) = \sum_{p=0}^{\infty} B_p^{(k)}(t)\,\boldsymbol{\phi}_p^{(k)}(\eta).$$

(This expansion is the counterpart of (3.7), an expansion of the $\mathbf{\Psi}_k(\eta,t)$ in terms of the $\boldsymbol{\phi}_p^{(k)}(\eta)$ with coefficients $A_p^{(k)}(t)$.)

We first record the fact that the forcing terms in the equations for the coefficients $B_p^{(k)}(t)$, using the unscaled $\mathbf{F}_k^{(l)}$ defined in (3.4), turn out to have a magnitude which is $O\left(\sum_{l=1}^{N} \mathbf{F}_k^{(l)} \Big/ \epsilon d_k\right)$, unless $\tau_0^{(k)} = o(1)$ in which case this magnitude is

$$O\left[\sum_{l=1}^{N} \mathbf{F}_k^{(l)} \Big/ \epsilon d_k \tau_0^{(k)}\right].$$

To see this, one must make an estimate with the aid of an integration by parts, as in Eckhaus (1965, §7). Inclusion of the factor $\tau_0^{(k)}$ is harmless if $\tau_0^{(k)} = O(1)$. Hence, if

$$\mathbf{F}_k^{(l)} \approx \epsilon d_k \tau_0^{(k)} \tag{4.3}$$

then the forcing terms in the equations for the $B_p^{(k)}(t)$ are $O(1)$. Hence the $\psi_k(\eta, t)$ should certainly be $O(1)$, when $\tau_0^{(k)} = O(1)$. Condition (4.3) also ensures that $\psi_k(\eta,t) = O(1)$ for $k \in J^*$, a k-region where $\tau_0^{(k)} \neq O(1)$, for then as in the parallel equation (3.8)

$$\psi_k(\eta,t) = B_0^{(k)}(t)\,\phi_0^{(k)}(\eta) + O(\epsilon^2).$$

The coefficient $B_0^{(k)}$ satisfies an equation which is analogous to (3.9) and which has an $O(1)$ solution. For all k, then, we must determine d_k so that (4.3) holds.

To see what (4.3) entails it is useful to write (3.4*b*) symbolically as

$$\mathbf{F}_k^{(l)} \approx \mathbf{\Phi}_0\mathbf{\Phi}_k + \sum_{k'=\Delta k}^{k-\Delta k} \mathbf{\Phi}_{k'}\mathbf{\Phi}_{k-k'} + \sum_{k'=\Delta k}^{\infty} \mathbf{\Phi}_{k'}\mathbf{\Phi}_{k+k'} \quad (k > 0). \tag{4.4}$$

The symbolic notation in (4.4) means that for $\epsilon \to 0$ the order of magnitude of the left-hand side equals the order of magnitude of the right-hand side. Expression (4.4) for $\mathbf{F}_k^{(l)}$ retains its essence, since the operations of differentiation, summing, and taking complex conjugates do not affect the order of magnitude estimates. With this symbolic notation, (4.4) also holds for $\mathbf{F}_0^{(l)}$. Substituting (4.1) into (4.4) we find for $k \geqslant 0$

$$(\epsilon d_k)^{-1}\mathbf{F}_k^{(l)} \approx \epsilon d_0 + \epsilon d_k^{-1} \sum_{k'=\Delta k}^{k-\Delta k} d_{k'} d_{k-k'} + \epsilon d_k^{-1} \sum_{k'=\Delta k}^{\infty} d_{k'} d_{k+k'}, \tag{4.5}$$

where $O(1)$ factors involving ψ's are omitted in our symbolic notation.

When $\tau_0^{(k)} \approx 1$ the left side of (4.5), by (4.3), is ≈ 1. Since $d_k \prec 1$, the sums on the right side of (4.5) are $\prec 1$. To obtain the proper contribution from the right side of (4.5) we must have

$$d_k \prec \epsilon \quad \text{when} \quad \tau_0^{(k)} \approx 1, \tag{4.6}$$

a result which we shall need later.

It is now not difficult to see that the ϵd_0 term can be omitted from the requirement on the d's given in (4.5). This follows from the fact that $O(\tau_0^{(k)}) > \epsilon^2$ for all k corresponding to unknown d_k, i.e. for $k \notin J$. Using (4.3), the left side of (4.5) is therefore greater in magnitude than ϵ^2 for $k \notin J$. Since $d_0 \prec \epsilon$ by (4.6), we have the desired result.

Equations (4.3) and (4.5) therefore imly that

$$d_k \approx \frac{\epsilon}{\tau_0^{(k)}} \max\Big[\sup_{0<k'<k} d_{k'} d_{k-k'};\quad \sup_{k'>0} d_{k'} d_{k+k'}\Big] \quad (k \notin J). \tag{4.7}$$

The sup symbol is to be interpreted as requiring the term of highest order of magnitude which is encounted as k' traverses the indicated intervals (in steps of width Δk). Derivation of (4.7) requires that the infinite sums in (4.5) be split into finite sums (whose magnitude is determined by their term of largest magnitude) plus remainders which can be made arbitrarily small by taking enough terms in the finite sums.

To begin our examination of how (4.7) restricts the d's, we consider the special case where k is within $O(\epsilon)$ of $3k_0$. Because of (4.2) and (4.6) the sup in (4.7) is attained from pairs of d's, one of whose subscripts refers to a wave-number in J. Thus $d_k \prec \epsilon^2$ for $k = 3k_0 + O(\epsilon)$.

Next, suppose that k is within $O(\epsilon)$ of $4k_0$. Proceeding as in the previous paragraph we observe that the terms from which the sup is to be selected are no larger in magnitude than $d_{k'} d_{k-k'}$ where $d_{k'} = O(1)$ [k' near k_0] and $d_{k-k'} = O(\epsilon^2)$ [k' near $3k_0$]. If $O(4k_0 - k) > \epsilon$, the sup will be smaller. To see this, consider for example the above-mentioned term $d_{k'} d_{k-k'}$. If $k - k'$ is within $O(\epsilon)$ of $3k_0$ then k' must be farther than $O(\epsilon)$ from k_0, with a resulting contribution to $d_{k'} d_{k-k'}$ which is less than $O(\epsilon)$.

Reasoning as in the above treatment of wave-numbers near $3k_0$ and $4k_0$ we see that there is a sequence of neighbourhoods, around mk_0, $m > 2$, where $d_k \prec \epsilon^{|m-1|}$. By similar reasoning one can determine restrictions on the d_k when $k - mk_0 = 0$ for all m, $m = 0, 1, 2, \ldots$. The first step in such reasoning shows that $d_k \prec \epsilon^2$ for such k. There is no need to carry the reasoning any further, for one can now assert that unless k is near 0, k_0, or $2k_0$ both free and forced contributions to $\mathbf{\Phi}_k$ are no larger in magnitude than $O(\epsilon^3)$ so that (Eckhaus 1965, chapter 8) these $\mathbf{\Phi}_k$ can be ignored in determination of the dominant terms in the equations for the $\mathbf{\Phi}_k$, k near 0, k_0, $2k_0$.

It remains to determine the consequences of (4.7) when k is near 0, k_0 and $2k_0$. We write

$$d_k(\epsilon) = \epsilon d_k^{(1)}(\epsilon) \quad \text{when} \quad O(k) < 1, \tag{4.8a}$$

and

$$d_k(\epsilon) = \epsilon d_k^{(2)}(\epsilon) \quad \text{when} \quad O(2k_0 - k) < 1. \tag{4.8b}$$

Considerations like those of the next to last paragraph show that

$$d_k^{(1)}(\epsilon) = O(1) \quad \text{if} \quad k \in I; \quad d_k^{(2)}(\epsilon) = O(1) \quad \text{if} \quad k \in Y. \tag{4.9}$$

Using (4.8), (4.9) and (4.2) it is not hard to deduce from (4.7) that

$$d_k^{(1)} = \sup_{k' \in J^*} d_{k'} d_{k+k'} \quad (k \in I^*); \tag{4.10}$$

$$d_k^{(2)} = \sup_{k' \in J^*} d_{k'} d_{k-k'} \quad (k \in Y^*); \tag{4.11}$$

$$d_k = \frac{\epsilon^2}{\tau_0^{(k)}} \max\{\sup_{k' \in I^*} d_{k'}^{(1)} d_{k-k'};\ \sup_{\substack{k' \in J^* \\ k' < k}} d_{k'} d_{k-k'}^{(1)};\ \sup_{k' \in I^*} d_{k'}^{(1)} d_{k+k'};\ \sup_{k' \in J^*} d_{k'} d_{k+k'}^{(2)}\} \quad (k \in J^*). \tag{4.12}$$

We can now set up a cyclic sequence of estimates for the d_k, $k \in I^*$, J^*, Y^*. We first observe that $\tau_0^{(k)}$ has the same order of magnitude as $\epsilon^2 + (k - k_0)^2$ when $O(k-k)_0 < 1$, for

$$\epsilon^2 + (k-k_0)^2 = O(\epsilon^2) \quad \text{when} \quad O(k - k_0) \leqslant \epsilon,$$

$$= O(k-k_0)^2 \quad \text{when} \quad \epsilon < O(k-k_0) < 1.$$

But, from (3.5), when $\epsilon < O(k - k_0) < 1$

$$\tau_0^{(k)} = O(k-k_0)^2 \quad \text{since} \quad O[(k-k_0)^2] > \epsilon^2 \quad \text{and} \quad O[(k-k_0)^2] > O[\epsilon^2(k-k_0)].$$

Using this, (4.9) and (4.12) allows us to write

$$d_k(\epsilon) \prec f(k,\epsilon) \quad \text{when} \quad O(k-k_0) < 1, \tag{4.13}$$

where
$$f(k,\epsilon) = \epsilon^2/[\epsilon^2+(k-k_0)^2], \tag{4.14}$$

since $d_k(\epsilon) \prec 1$.

Starting with (4.13) we obtain successively more accurate estimates of d_k for $k \in I^*$, J^*, Y^* by cycling through (4.10) to (4.12). Thus, (4.10) and (4.13) imply

$$d_k^{(1)}(\epsilon) \prec \max_{k' \in J^*} f(k',\epsilon) f(k'+k,\epsilon) \quad (k \in I^*).$$

From figure 4 it is clear that the max requires either k' or $k+k'$ to be in J. If $k' = k_0 + O(\epsilon)$

$$f(k',\epsilon) f(k'+k,\epsilon) = \frac{\epsilon^2}{\epsilon^2+[O(\epsilon)]^2} \frac{\epsilon^2}{\epsilon^2+[k+O(\epsilon)]^2} \approx g_1(k,\epsilon), \tag{4.15}$$

where
$$g_1(k,\epsilon) = \epsilon^2/[\epsilon^2+k^2].$$

Thus
$$d_k^{(1)}(\epsilon) \prec g_1(k,\epsilon) \quad (k \in I^*). \tag{4.16}$$

Similarly,
$$d_k^{(2)}(\epsilon) \prec g_2(k,\epsilon) = \epsilon^2/[\epsilon^2+(2k_0 \mp k)^2] \quad (k \in Y^*), \tag{4.17}$$

where the choice of sign is immaterial. Using (4.13), (4.16) and (4.17) successively in (4.10), (4.11) and (4.12), respectively, we deduce

$$d_k^{(1)}(\epsilon) \prec g_1^M(k,\epsilon), \quad k \in I^*; \quad d_k(\epsilon) \prec f^M(k,\epsilon), \quad k \in J^*;$$
$$d_k^{(2)}(\epsilon) \prec g_2^M(k,\epsilon), \quad k \in Y^*; \tag{4.18}$$

where M is any positive integer.

We are now ready to obtain our final estimates for the d_k. When

$$k \in J, \quad k-k_0 = O(\epsilon) \quad \text{so} \quad f(k,\epsilon) = O(1).$$

Thus (4.18) shows that $d_k(\epsilon) = O(1)$ for $k \in K$. When $k \in J^*$, $k > O(k-k_0) > \epsilon$ so $f(k,\epsilon) = O[\epsilon^2/(k-k_0)^2]$. Thus (4.18) implies that $d_k(\epsilon)$ goes to zero with ϵ faster than any power of $\epsilon/(k-k_0)$ for $k \in J^*$. In particular $d_k(\epsilon)$ goes to zero faster than any power of ϵ if $k-k_0 = O(\epsilon^\alpha)$, $0 < \alpha < 1$. Of course, $d_k(\epsilon) = o(1)$ for $k \in J^*$. Similar results hold for I^* and Y^*. Thus (to sum up) for (4.10), (4.11) and (4.12) to hold it is necessary that

$$d_k(\epsilon) \prec \left[\frac{\epsilon}{k-k_0}\right]^{2M_0}, \quad k \in J^*; \quad d_k^{(1)}(\epsilon) \prec \left[\frac{\epsilon}{k}\right]^{2M_1}, \quad k \in I^*;$$
$$d_k^{(2)}(\epsilon) \prec \left[\frac{\epsilon}{2k-k_0}\right]^{2M_2}, \quad k \in Y^*; \tag{4.19}$$

where M_0, M_1 and M_2 are arbitrary positive integers. Moreover, the requirements of (4.19) certainly do not contradict the specifications of (4.10), (4.11) and (4.12). Note also that (4.19) implies that $d_k(\epsilon)$, $d_k^{(1)}(\epsilon)$ and $d_k^{(2)}(\epsilon)$ must be $o(1)$ for $k \in J^*$, I^*, Y^* respectively, which is equivalent to what was assumed in (3.6) when $k \in J^*$, I^*, Y^*.

Let us look back at what we have accomplished. In the derivation of the amplitude equations (3.11) we assumed that the forced modes $\boldsymbol{\Psi}_k$ defined in (3.6) were $O(1)$ uniformly in time, $k \notin J$. The reasoning of the present section shows that this assumption is valid provided that the scales of the forced modes decrease rapidly according to (4.19); so that (4.10), (4.11) and (4.12) can be satisfied. This means that initial values must be restricted to those of the form pictured in figure 3. [Because modes with wave-numbers k not in J initially decay exponentially, we conjecture that all modes could be assigned $O(\epsilon)$ initial conditions. After an

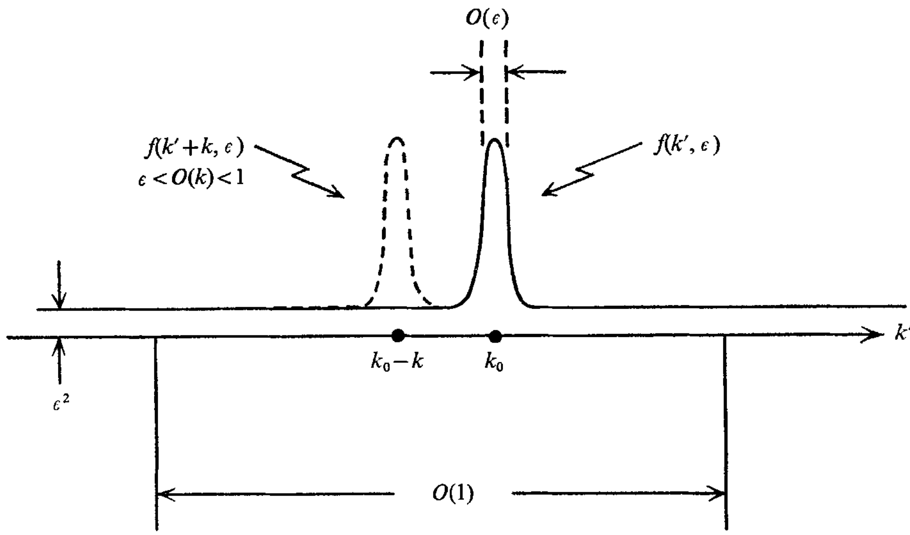

FIGURE 4. Graphs useful in establishing (4.15). The function of f is defined in (4.14).

'initial layer' in time $O(\epsilon^2)$ we expect that the exponential decay (according to linear theory) would reduce the magnitude of the modes so that the restrictions we require are satisfied. For application of the initial layer concept to periodic disturbance, see Matkowsky (1970).]

In the present section (and in §3) we assumed that the $\psi_k(\eta, t)$ are $O(1)$ uniformly in time, $k \in J$. But according to (3.8) and (3.10a) these $\psi_k(\eta, t)$ have the same magnitude as the $A_0^{(k)}(t)$ which are determined by the system (3.11). In the next section we complete the demonstration of consistency by showing that the solutions of (3.11) are $O(1)$ uniformly in time.

5. Discussion of the amplitude equations

To recapitulate, we have considered the development in time of a perturbation $\mathbf{\Phi}'$ given below. The unperturbed basic flow is characterized by the parameter R. The analysis has been simplified by the assumption that the parameter ϵ is small, where $\epsilon^2 = \epsilon^2(k_0, R) = |\tau_0^{(k_0)}(R)|$. Here $\tau_0^{(k_0)} = \operatorname{Re} \mu_0^{(k_0)}$ where $\mu_0^{(k_0)}$ is the growth rate for a mode with wave-number k_0, and k_0 is a wave-number within $O(\epsilon^2)$ of the critical wave-number k_c. Each mode considered in the analysis has a wave-number k which can be written as $k = m\Delta k$ for some integer m. We have set $\Delta k = \sigma\epsilon$ where $\sigma = O(1)$ (which does not preclude the possibility that $\sigma = o(1)$).

The Fourier analysis of the perturbation $\boldsymbol{\Phi}'$ is given by (3.1):

$$\boldsymbol{\Phi}'(\xi,\eta,t) = \sum_{m=-\infty}^{\infty} \boldsymbol{\Phi}_{m\Delta k}(\eta,t)\, e^{-im\Delta k\xi}.$$

In (3.6) we proposed a certain scaling of the coefficients $\boldsymbol{\Phi}_{m\Delta k}$. In §§ 3 and 4 we showed that if this scaling is valid initially then it is uniformly valid in time. Of primary interest are the most dangerous modes $\boldsymbol{\Phi}_k(\eta,t)$ with wave-numbers in the set $J = \{k \mid k-k_0 = O(\epsilon)\}$. In our analysis we assume that the most dangerous modes initially have magnitudes $O(\epsilon)$. It then turns out that there is a self-consistent scaling with the first harmonics of the most dangerous modes, the mean motion perturbation, and other first interactions all $O(\epsilon^2)$, and with other modes of higher order. Further, the first harmonics and mean motion can be expressed in terms of the most dangerous modes (see § 3 and appendix A). The denumerable set of most dangerous modes is given to lowest order by (3.12):

$$\boldsymbol{\Phi}_{k_0+n\sigma\epsilon}(\eta,t) = \epsilon[A_n(t)\,\boldsymbol{\phi}_0^{(k_0+n\sigma\epsilon)}(\eta)] + O(\epsilon^2) \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Here $\boldsymbol{\phi}_0^{(k)}(\eta)$ is the eigenfunction of linear theory defined above (2.7). The functions $A_n(t)$ satisfy the amplitude equations (3.11)

$$dA_n/dt + \mu_0^{(n)} A_n = -\epsilon^2\beta_c \sum_{p\in J} A_p \sum_{m\in J} \bar{A}_m A_{n+m-p} + O(\epsilon^3) \quad (n = 0, \pm 1, \pm 2, \ldots),$$

where the summation notation means a sum over all of the most dangerous modes. In these amplitude equations, $\beta_c = \beta^{(k_c)}$ is the Landau constant for a disturbance having the critical wave-number [see (2.13) and the remarks following it] while $\mu_0^{(n)}$ is an abbreviation for $\mu_0^{(k_0+n\sigma\epsilon)}$.

The most interesting consequences of our analysis follow from (3.11). It is consistent with our previous analysis to expand $\tau_0^{(n)}$ and $\nu_0^{(n)}$ about k_0 for a fixed R with $R-R_c = O(\epsilon^2)$ and to retain only terms through $O(\epsilon^2)$. First,

$$\nu_0^{(n)} = \nu_0 + \epsilon\sigma n\nu_1 + \epsilon^2\sigma^2 n^2\nu_2 + O(n^3\sigma^3\epsilon^3), \tag{5.1}$$

where the parameter ν_1 is the group speed of the family of waves and $\nu_2 = \frac{1}{2}\partial^2\nu/\partial k^2$ evaluated at k_0(or k_c) is one-half of the rate of change of group speed with wave-number. Next, making use of (3.5a), (3.5b) and the fact that $\epsilon^2 = |\tau_0^{(0)}(R)|$ we can write

$$\begin{aligned}\tau_0^{(n)} &= \tau_0^{(0)} + \epsilon^2\sigma^2 n^2\alpha^2 + O(\epsilon^3 n\sigma)\\ &= \epsilon^2\left[\alpha^2\sigma^2 n^2 + \frac{\tau_0^{(0)}}{|\tau_0^{(0)}|}\right] + O(\epsilon^3 n\sigma),\end{aligned} \tag{5.2}$$

where
$$\alpha = \left(\frac{1}{2}\frac{\partial^2\tau_0^{(k)}}{\partial k^2}\bigg|_{k=k_c,\,R=R_c}\right)^{\frac{1}{2}}. \tag{5.3}$$

Within the error in our analysis the term $\partial^2\tau_0^{(k)}/\partial k^2$ in (3.5a) can be evaluated at either (k_0, R) or at (k_c, R_c), and is necessarily non-negative for the problems under consideration since (k_c, R_c) is a minimum point on the neutral curve. The parameter α is a measure of the change in growth rate with changing wave-number for fixed R. As we have indicated earlier $\sigma = \Delta k/\epsilon$, and hence σ measures the step size between the wave-numbers of successive modes relative to the width of the band of wave-numbers of the most dangerous modes. We emphasize that $\epsilon\sigma n = O(\epsilon)$ for all n.

Finally, letting

$$\tau = \epsilon^2 t, \quad A_n(t) = \exp\left[-i(\nu_0+\epsilon\sigma n\nu_1)t\right]a_n(\epsilon^2 t), \tag{5.4}$$

we obtain from (3.11)

$$da_n/d\tau + \left(\alpha^2\sigma^2 + \frac{\tau_0^{(0)}}{|\tau_0^{(0)}|} + i\sigma^2 n^2\nu_2\right)a_n = -\beta_c \sum_{p\in J}\sum_{m\in J} a_p\bar{a}_m a_{n+m-p} + O(\epsilon, \epsilon\sigma n a_n, \epsilon n^3\sigma^3 a_n), \tag{5.5}$$

for $n = 0, \pm 1, \pm 2, \ldots$ with $k_0 + n\sigma\epsilon$ in J. In (5.5) we have explicitly shown the order of the various error terms which arise from the expansion of $\mu_0^{(n)}$. In the following the error term will not always be recorded. Also in the remainder of this section we shall take $R > R_c$, so that $\tau_0^{(0)}/|\tau_0^{(0)}| = -1$; in a study of subcritical instabilities this ratio should be replaced by 1.

We write

$$a_n = |a_n|\exp(i\theta_n), \quad \beta_c = |\beta_c|\exp(i\chi),$$

and define the 'energy' E by

$$E = \tfrac{1}{2}\sum_{n\in J}|a_n|^2. \tag{5.6}$$

It is a straightforward matter to verify that

$$dE/d\tau = \sum_{n\in J}(1-\alpha^2\sigma^2 n^2)|a_n|^2 - (\cos\chi)|\beta_c|\sum_{n\in J}(C_n^2+S_n^2), \tag{5.7}$$

where

$$C_n = \sum_{p\in J}|a_p||a_{p+n}|\cos(\theta_p-\theta_{p+n}), \quad S_n = \sum_{p\in J}|a_p||a_{p+n}|\sin(\theta_p-\theta_{p+n}). \tag{5.8}$$

If the real part of the Landau constant β_c, $|\beta_c|\cos\chi$, is positive then the second term on the right-hand side of (5.7) is negative, and consequently the non-linear terms will have a stabilizing effect.† Thus, if non-linear effects act to stabilize a small two-dimensional disturbance composed primarily of a single mode with a near-critical wavelength (as they are well known to do in the Taylor and Bénard problems, where β_c is real and positive) then the non-linear terms also stabilize small disturbances composed primarily of a number of modes with wave-numbers near critical.

Clearly an analysis of the infinite set of non-linear ordinary differential equations (5.5) for the amplitude functions $a_n(\tau)$ is a formidable task. However, considerable insight can be gained by the introduction of a special Fourier transform of the $a_n(\tau)$. Such a transform allows the replacement of (5.5) by a single non-linear partial differential equation; and moreover in certain cases the perturbation $\mathbf{\Phi}'$, correct through terms $O(\epsilon)$, can be expressed directly in terms of the transform function.

We thus define the complex-valued transform function Z by

$$Z(\omega,\tau) = \sum_{n\in J} a_n(\tau)\,e^{-i\sigma n\omega}. \tag{5.9a}$$

† Equation (5.7) has been derived using (5.5) in which the growth rate has been expanded. However, one can work directly with (3.11); only the linear terms are altered and the non-linear terms are stabilizing as above.

By definition, Z is a periodic function of ω:

$$Z(\omega+(2\pi j/\sigma),\tau) = Z(\omega,\tau), \tag{5.9b}$$

where j is a positive integer. Further,

$$(2\pi j/\sigma)\,a_n(\tau) = \int_0^{(2\pi j)/\sigma} e^{in\sigma\omega} Z(\omega,\tau)\,d\omega. \tag{5.10}$$

It is easy to see that (5.5) is equivalent to

$$\frac{\partial Z}{\partial \tau} = (\alpha^2+i\nu_2)\frac{\partial^2 Z}{\partial \omega^2} + Z(1-\beta_c|Z|^2) + O(\epsilon, \epsilon\,\partial Z/\partial\omega, \epsilon\,\partial^3 Z/\partial\omega^3), \tag{5.11}$$

with the periodicity condition (5.9b). Note that

$$E(\tau) = \frac{\sigma}{4\pi}\int_0^{2\pi/\sigma} |Z(\omega,\tau)|^2\,d\omega. \tag{5.12}$$

To illustrate the use of the transform Z, we shall make a precise statement concerning the global stability of solutions of the system (5.5). (This can also be done directly, using (5.7).) From (5.11)

$$Z\frac{\partial \bar{Z}}{\partial \tau} + \bar{Z}\frac{\partial Z}{\partial \tau} = \alpha^2\left[Z\frac{\partial^2 \bar{Z}}{\partial \omega^2} + \bar{Z}\frac{\partial^2 Z}{\partial \omega^2}\right] - i\nu_2\left[Z\frac{\partial^2 \bar{Z}}{\partial \omega^2} - \bar{Z}\frac{\partial^2 Z}{\partial \omega^2}\right] + 2|Z|^2(1-\operatorname{Re}\beta_c|Z|^2).$$

Integrating with respect to ω, by parts on the terms in the square brackets, we obtain

$$\frac{2\pi}{\sigma}\frac{dE}{d\tau} = -\alpha^2\int_0^{2\pi/\sigma}\left|\frac{\partial Z}{\partial \omega}\right|^2 d\omega - \int_0^{2\pi/\sigma}[(\operatorname{Re}\beta_c)|Z|^4 - |Z|^2]\,d\omega,$$

where we have used (5.12). But according to a version of the Schwarz inequality

$$\frac{2\pi}{\sigma}\int_0^{2\pi/\sigma}|Z|^4\,d\omega \geqslant \left[\int_0^{2\pi/\sigma}|Z|^2\,d\omega\right]^2.$$

Thus if $\operatorname{Re}\beta_c > 0$, then

$$(dE/d\tau) < 0 \quad \text{if} \quad E > (2\operatorname{Re}\beta_c)^{-1}. \tag{5.13}$$

Using the relation between a_n and A_n given in (5.4) and assuming that

$$\operatorname{Re}\beta_c = O(1),$$

we conclude that the functions $A_n(t)$ are $O(1)$ uniformly in time if they are initially $O(1)$. In accord with our remarks at the end of § 4, this result completes our demonstration that it is consistent to assume that the initial scaling (3.6) is uniformly valid in time. In addition, from (5.13) we note that within the limits of this theory, a steady equilibrium state must have energy E less than $(2\operatorname{Re}\beta_c)^{-1}$.

Let us examine the relation between the perturbation $\mathbf{\Phi}'$ and the transform function Z. By approximating $\boldsymbol{\phi}_0^{(k_0+n\sigma\epsilon)}(\eta)$ by $\boldsymbol{\phi}_0^{(k_0)}(\eta)$ we can use (3.12) to write

$$\mathbf{\Phi}'(\xi,\eta,t) = 2\epsilon\operatorname{Re}[\boldsymbol{\phi}_0^{(k_0)}(\eta)\sum_{n\in J} e^{-i(k_0+n\sigma\epsilon)\xi}A_n(t)] + O(\epsilon^2). \tag{5.14}$$

With (5.4) this becomes

$$\Phi'(\xi,\eta,t) = 2\epsilon\operatorname{Re}[\boldsymbol{\phi}_0^{(k_0)}(\eta)\,e^{-i(k_0\xi+\nu_0 t)}\sum_{n\in J} e^{-i\epsilon\sigma n(\xi+\nu_1 t)}a_n(\epsilon^2 t)] + O(\epsilon^2). \tag{5.15}$$

At this point it is necessary to consider the cases $\nu = 0$ and $\nu \neq 0$ separately. We begin with the case $\nu = 0$ (axisymmetric Taylor problem, Bénard problem). In this case, using the definition (5.9a), and remembering that $\nu_1 = 0$ if $\nu = 0$, we see that (5.15) becomes

$$\boldsymbol{\Phi}'(\xi,\eta,t) = 2\epsilon\,\mathrm{Re}\,[\boldsymbol{\phi}_0^{(k_0)}(\eta)\,e^{-ik_0\xi}Z(\epsilon\xi,\epsilon^2 t)] + O(\epsilon^2), \tag{5.16}$$

so that $\boldsymbol{\Phi}'$ can be expressed directly in terms of Z without the intervention of the modal amplitudes $a_n(t)$. If Z were independent of ξ then (5.16) would provide the appropriate leading term in a non-linear analysis of the development of a disturbance which is periodic in ξ with the wave-number k_0. (Compare the limiting formula (2.15).) As written, (5.16) can therefore be regarded as taking into account possible spatial modulations of this disturbance's amplitude. The presence of the ϵ factors in the arguments of Z means that the modulations vary spatially with an $O(\epsilon^{-1})$ length scale and that the shape of the modulating function Z (or envelope) changes on an $O(\epsilon^{-2})$ time scale.

Segel (1969) and Newell & Whitehead (1969) studied such slow modulations in the free–free Bénard problem using formal multiple scale techniques. Not surprisingly their direct approach required considerably less computational effort than our approach. In this special case (of course $\nu = 0$ for the Bénard problem) our equation (5.11) is in agreement with their result.

A question of fundamental importance is that of wave-number selection in a competition between the most dangerous modes. Snyder (1969), for example, has demonstrated that for the Taylor problem, steady axisymmetric flows periodic in the axial direction with different wave-numbers can be obtained at the same value of R by changing the initial conditions. Also see Chen & Whitehead (1968) for similar results on the Bénard problem and Coles (1965) for the non-axisymmetric Taylor problem. Suppose that as $t \to \infty$,

$$A_n(t) \to 0 \quad \text{for} \quad n \neq p \quad \text{and} \quad A_p(t) \to A_p^{(e)}.$$

Then it is easily seen that the perturbation tends to become spatially periodic with wave-number $k_0 + p\sigma\epsilon$. Wave-number selection has taken place. If selection from the most dangerous modes always occurs then the only stable steady solutions for R slightly greater than R_c will be spatially periodic with wave-numbers near k_c.

A full investigation of wave-number selection is well beyond the scope of this paper; however, we shall make one contribution to the matter. Our investigation has shown (in general) that if at least two of the most dangerous modes are initially excited at an $O(\epsilon)$ level then a multi-modal solution will exist for a time. A key question is, can a multi-modal solution persist for all time, or will all but one of the amplitudes A_n decay to zero. We shall now show that for the case $\nu = 0$, $\beta_c > 0$ there is a steady solution of (5.11). Thus the set of steady multi-modal solutions is non-empty in this case. On the other hand, the solution in question will be seen to be unstable, so whether or not selection takes place remains open. For recent work on this question see Ponomarenko (1968) and Newell *et al.* (1970).

To obtain the steady solution of (5.11) with $\nu = 0$, $\beta_c > 0$ we set

$$Z = (\beta_c)^{-\frac{1}{2}}\,r(x)\,e^{i\theta(x)}; \quad r(x) > 0, \quad x = \omega\alpha^{-1}. \tag{5.17}$$

We see that (5.11) with $\nu_2 = 0$ is satisfied if

$$r'' - c_0^2 r^{-3} + r(1-r^2) = 0, \tag{5.18}$$

where
$$\theta(x) = c_0 \int_0^x \frac{1}{r^2(\lambda)}\, d\lambda, \tag{5.19}$$

and c_0 is a constant. Because of the periodicity condition (5.9*b*), the function r and the constant c_0 must satisfy

$$r\left(x + \frac{2\pi m}{\alpha\sigma}\right) = r(x), \quad c_0 \int_x^{x+2\pi m/\alpha\sigma} \frac{1}{r^2(\lambda)}\, d\lambda = 2n\pi \quad (n = 0, 1, 2, \ldots), \tag{5.20}$$

for some positive integer m. The simplest case is when $n = 0$, $c_0 = 0$ and $\theta(x) = 0$. Then r satisfies an elliptic equation with solution

$$r(x) = a\, sn\, bx, \quad a^2 = 2k^2(1+k^2)^{-1}, \quad b = (1+k^2)^{-2}. \tag{5.21}$$

The modulus k of the elliptic function is determined by

$$4K(k)/b = 2\pi m/\alpha\sigma,$$

where K is the quarter-period of the elliptic function. See Arscott (1964, appendix C).

To examine the stability of the solution (5.21) we write

$$Z(\omega, \tau) = (\beta_c)^{-\frac{1}{2}} r(\alpha^{-1}\omega) + Z_1(\omega, \tau). \tag{5.22}$$

We substitute into (5.11) and linearize. Remembering that we are considering the case $\nu = 0$ and writing

$$Z_1(\omega, \tau) = \exp(s\tau)\, \zeta(y) \quad (y = b\alpha^{-1}\omega), \tag{5.23}$$

we find that ζ must satisfy the Lamé equation

$$d^2\zeta/dy^2 + \zeta[H - 6k^2 sn^2 y], \quad \text{where} \quad H = (1+k^2)(1-s). \tag{5.24}$$

From Arscott (1964, p. 205), supplying a missing factor of two, if

$$H = 2(1+k^2) - 2(1-k^2k'^2)^{\frac{1}{2}} \quad (k'^2 = 1-k^2), \tag{5.25}$$

then (5.23) is satisfied by

$$\zeta(y) = sn^2 y - 3k^{-2}[1 + k^2 - (1-k^2k'^2)^{\frac{1}{2}}]. \tag{5.26}$$

When H has the value given by (5.25) it can be seen that the growth rate s is positive. Hence Z_1 is unbounded with increasing time and the multi-modal equilibrium solution (5.21) is unstable.

Segel (1969) found that in two-dimensional Bénard convection the steady solution (5.21) provided an amplitude modulation which was sufficient to 'fit' the steady supercritical solution which is appropriate for unbounded horizontal layers between vertical walls. It was necessary to require that r vanish at the end points of its domain of definition, so that the boundary conditions at the walls could be satisfied. If the walls were far enough apart, r might also vanish in the interior, giving 'imaginary wall solutions'. The instability of the imaginary wall solutions can be demonstrated (see Segel 1970) but the modulation with no interior zeros appears to be stable. (The perturbation (5.26) is not admissible in this case, as it does not satisfy the boundary conditions.) Newell & Whitehead (1969) also discussed aspects of the solutions of (5.11) and their stability in the Bénard problem.

We now turn to the case $\nu \neq 0$. We shall find that the situation is considerably different from the case $\nu = 0$. Use of the transform Z no longer leads to easily interpreted results for it is now impossible to cast (5.15) into a form so that $\mathbf{\Phi}'$ is expressed directly in terms of Z. To determine the leading terms in $\mathbf{\Phi}'$ one must solve the partial differential equation (5.11) for the function Z, compute the a_n by the inverse transform formula (5.10), and then employ the superposition (5.15). However, by proceeding in a slightly different manner, we can obtain a result similar to (5.16) in the case $\nu \neq 0$.

Let us reconsider the amplitude equations (3.11) with $\nu_0^{(n)}$ and $\tau_0^{(n)}$ given by (5.1) and (5.2); we have

$$dA_n/dt + [i\nu_0 + \epsilon i\sigma n\nu_1 + \epsilon^2(\alpha^2\sigma^2 n^2 - 1 + i\sigma^2 n^2\nu_2)]\,A_n$$
$$= -\epsilon^2\beta_c \sum_{p\in J}\sum_{m\in J} A_p \bar{A}_m A_{n+m-p} + O(\epsilon^3, \epsilon^3 n\sigma A_n, \epsilon^3 n^3\sigma^3 A_n). \quad (5.27)$$

Now let
$$A_n(t) = e^{-i\nu_0 t} a_n^*(t), \quad Z^*(\omega, t) = \sum_{n\in J} e^{-in\sigma\omega} a_n^*(t), \quad (5.28)$$

where the a_n^* can be expressed in terms of Z^* in the usual way. From (5.14) we have

$$\mathbf{\Phi}'(\xi,\eta,t) = 2\epsilon\,\mathrm{Re}\,[\boldsymbol{\phi}_0^{(k_0)}(\eta)\,e^{-i(k_0\xi+\nu_0 t)} \sum_{n\in J} e^{-in\sigma\epsilon\xi} a_n^*(t)] + O(\epsilon^2)$$
$$= 2\epsilon\,\mathrm{Re}\,[\boldsymbol{\phi}_0^{(k_0)}(\eta)\,e^{-i(k_0\xi+\nu_0 t)} Z^*(\epsilon\xi, t)] + O(\epsilon^2). \quad (5.29)$$

Thus the leading term in the expression for $\mathbf{\Phi}'$ has been expressed in terms of the function Z^*.

The equation for Z^* is obtained from (5.27) making use of (5.28):

$$\frac{\partial Z^*}{\partial t} = \epsilon\nu_1 \frac{\partial Z^*}{\partial \omega} + \epsilon^2(\alpha^2 + i\nu_2)\frac{\partial^2 Z^*}{\partial \omega^2} + \epsilon^2 Z^*(1 - \beta_c|Z^*|^2) + O\left(\epsilon^3, \epsilon^3\frac{\partial Z^*}{\partial \omega}, \epsilon^3\frac{\partial^3 Z^*}{\partial \omega^3}\right). \quad (5.30)$$

In the case $\nu_1 = 0$ and with the change of variables $\tau = \epsilon^2 t$ (5.20) reduces to (5.11) with $\nu_2 = 0$. Thus (5.11) is a special case of the present equation. To interpret (5.30) we first note from (5.29) that Z^* can be regarded as the envelope of a wave train with carrier whose wave-number and frequency are k_0 and ν_0, respectively. The $\partial Z^*/\partial\omega$ term reflects a tendency for the envelope to move at the group speed ν_1, for in co-ordinates moving with speed ν_1 in the negative ξ direction this term disappears (see below). The term $\partial^2 Z^*/\partial\omega^2$ is of diffusion type with complex 'diffusivity' $\alpha^2 + i\nu_2$ and can be traced to the n^2 terms in (5.27). Note that ν_2 is non-zero when waves of different length have different speeds, so the ν_2 contribution is due to dispersion. The term proportional to Z^* results in a growth of Z^* with time, reflecting an underlying instability of the basic flow. The term $Z^*|Z^*|^2$ gives rise to a non-linear modification of this growth.

Since ϵ appears in (5.30) it is clear that Z^* is not simply a function of ω and t, but also depends on ϵ: $Z^* = Z^*(\omega, t; \epsilon)$. Further, since ϵ^2 multiplies $\partial^2 Z^*/\partial\omega^2$, (5.30) is of the singular perturbation type. Thus the first approximation to the solution Z^* that we shall construct will depend on the values of the arguments $\omega = \epsilon\xi$ and t that are of interest. We therefore introduce the family of transformations

$$\xi = \epsilon^{-a}\omega^* \quad \text{or} \quad \omega = \epsilon^{1-a}\omega^*, \quad t = \epsilon^{-b}\tau^*, \tag{5.31}$$

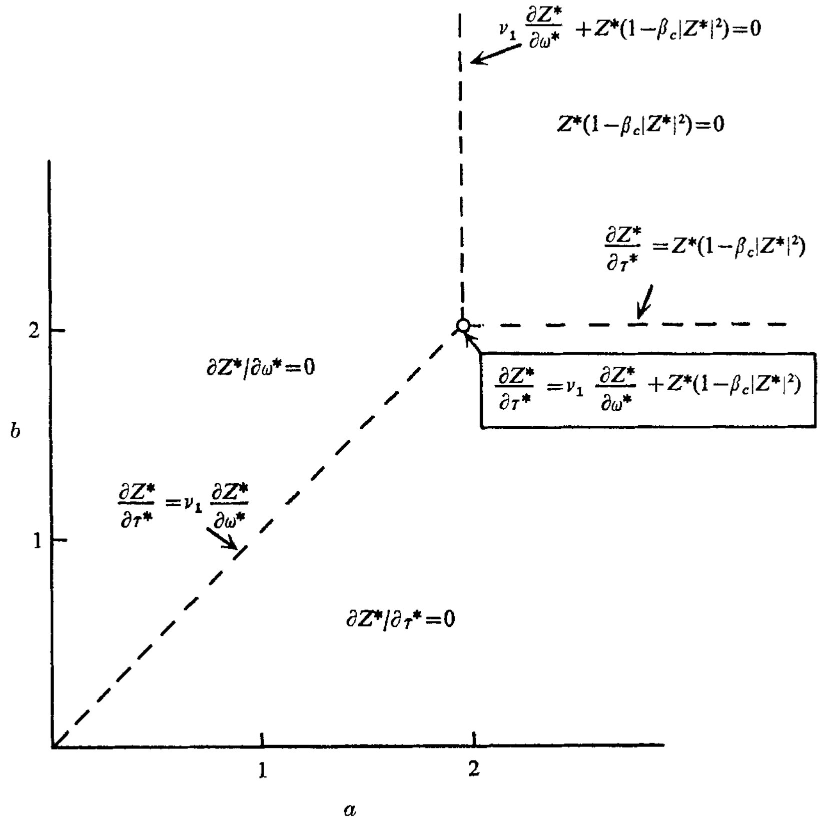

FIGURE 5. Limiting forms of (5.30) for $\xi = O(\epsilon^{-a})$ and $t = O(\epsilon^{-b})$ for different values of a and b.

with $a \geqslant 0, b \geqslant 0$. With the understanding that Z^* has been redefined in terms of the new variables τ^* and ω^*, we have from (5.30)

$$\epsilon^b \frac{\partial Z^*}{\partial \tau^*} = \epsilon^a \nu_1 \frac{\partial Z^*}{\partial \omega^*} + \epsilon^{2a}(\alpha^2 + i\nu_2)\frac{\partial^2 Z^*}{\partial \omega^{*2}} + \epsilon^2 Z^*(1-\beta_c|Z^*|^2) + O\left(\epsilon^3, \epsilon^{a+2}\frac{\partial Z^*}{\partial \omega^*}, \epsilon^{3a}\frac{\partial^3 Z^*}{\partial \omega^3}\right). \tag{5.32}$$

The limiting forms of (5.32) for different values of a and b are depicted in figure 5. We note that we must restrict a so that $a > 0$, for if $a = 0$, the error term in (5.32) which arises from the expansion of $\nu_0^{(n)}$ is of the same order as terms retained. For $a = 0$ it is not permissible to expand $\nu_0^{(n)}$ as in (5.1); however in this case the non-linear terms on (5.32) are negligible and the equation is of little interest.

As can be seen from figure 5, a distinguished limit occurs for $a = 2$, $b = 2$; that is $t = O(\epsilon^{-2})$. All other equations can be obtained from the equation for $a = b = 2$ by rescalings. Thus the equation

$$\frac{\partial Z_1^*}{\partial \tau^*} = \nu_1 \frac{\partial Z_1^*}{\partial \omega^*} + Z_1^*(1-\beta_c|Z_1^*|^2), \tag{5.33}$$

with

$$\tau^* = \epsilon^2 t, \quad \omega^* = \epsilon^2 \xi, \tag{5.34}$$

is a fundamental equation to be studied. The subscript 1 has been introduced to indicate that Z_1^* is the first approximation to Z^* in the limit $\epsilon \to 0$ with τ^* and ω^* fixed.

In this case we have from (5.29) that

$$\mathbf{\Phi}'(\xi, \eta, t) = 2\epsilon \operatorname{Re}[\boldsymbol{\phi}_0^{(k_0)}(\eta)\, e^{-i(k_0\xi+\nu_0 t)} Z_1^*(\epsilon^2\xi, \epsilon^2 t)] + O(\epsilon^2). \tag{5.35}$$

Equation (5.34) is equivalent to a fundamental equation derived independently by Stewartson & Stuart (1971) in their analysis of the non-linear instability of a wave system in plane Poiseuille flow. It is immediately evident that under the change of variables $\chi = \omega^* + \nu_1 \tau^* = \epsilon^2(\xi + \nu_1 t)$, (5.33) takes the form

$$\partial Z_1^*/\partial \tau^* = Z_1^*(1-\beta_c|Z_1^*|^2), \tag{5.36}$$

where Z_1^* is now regarded as a function of τ^* and the parameter χ. Equation (5.36) has the form of the classical Landau equation with χ as a parameter, and can be solved exactly. Alternatively, τ^* can be eliminated to obtain an equation for Z_1^* as a function of ω^* and the variable χ. Also with reference to the paper of Stewartson & Stuart, we note that their equation (4.10) in which the parameter $\xi + \nu_1 t$ is regarded as being variable rather than a constant, can be obtained from (5.30) by making the change of variables

$$\tau^* = \epsilon^2 t, \quad X = \epsilon(\xi + \nu_1 t), \quad \omega = \epsilon\xi, \tag{5.37}$$

which yields at lowest order

$$\frac{\partial Z_1^*}{\partial \tau^*} = (\alpha^2 + i\nu_2)\frac{\partial^2 Z_1^*}{\partial X^2} + Z_1^*(1-\beta_c|Z_1^*|^2). \tag{5.38}$$

We see, then, that our results, when comparable with those of Stewartson & Stuart, are in agreement with theirs. The reader is urged to consult the valuable paper of these two authors for a discussion which is closely related and complementary to the material of the present paper. A full treatment of (5.30) is beyond the scope of the present paper; however, we note that it would be of considerable interest to study the various degenerations of (5.30) and their matchings with the goal of determining the development with time of an initial disturbance.

Consider now the implications of the periodicity condition in ω on Z^*. From the definition (5.28) of Z^* it is clear that

$$Z^*(\omega + (2\pi/\sigma)j, t) = Z^*(\omega, t), \tag{5.39}$$

for some positive integer j. On the other hand, under the different limiting procedures described above we will obtain different equations for first approximations to Z^*. We require that these first approximations be periodic in ω, but there is an additional restriction. Suppose, for example, that using a certain limit we find a first approximation to the solution Z^* which has period T in ω. Then we must require for consistency as $\epsilon \to 0$ that $T/(2\pi/\sigma) = O(1)$ (which includes the possibility $T\sigma = o(1)$). This is necessary in order that the periodic solution of the differential equation can be fitted into the required period of the transform. Now consider the different limiting cases given above. The period of the solution of a limiting equation will be some number, say M, since ϵ does not appear in the limiting equation. Since $\omega = \epsilon^{1-a}\omega^*$ it follows that the period in ω is $T = \epsilon^{1-a}M$ Now the condition $T\sigma = O(1)$ yields

$$\epsilon^{1-a}\sigma = O(1). \tag{5.40}$$

Thus for $a \leqslant 1$ no restriction on σ results; however for $a > 1$ we have $\sigma = O(\epsilon^{a-1})$. In particular, for the distinguished limit $a = 2$, $b = 2$ we obtain $\sigma = O(\epsilon)$. Recalling that $\sigma = \Delta k/\epsilon$ this means that Δk must be chosen $O(\epsilon^2)$ or smaller; that is, the band of most dangerous wave-numbers which has width $O(\epsilon)$ must be covered by modes with wave-numbers separated by at most a width $O(\epsilon^2)$.

Turning from the details of the analysis, we point out that a fluid-mechanical situation which may give rise to the slightly non-conservative dispersive waves which we have been discussing is the super-critical flow between counter-rotating circular cylinders. When the gap between the cylinders is small and the ratio of the outer to inner angular speeds is less than about -1, Krueger, Gross & DiPrima (1966) have shown that the critical disturbance is a non-axisymmetric mode. The corresponding growth rate is complex and the Landau constant is complex and has a positive real part. Non-linear analysis of disturbances which are periodic in both azimuthal and axial directions predicts a final state which takes the form of a spiral vortex travelling in both the axial and azimuthal directions (DiPrima & Grannick 1970). Of course our two-dimensional analysis does not directly apply here, but the fact that dependence on the third (azimuthal) variable must be periodic encourages the belief that extension of our analysis would lead to essentially the same kinematic behaviour as has just been described.

Another fluid-mechanical problem of interest is the stability of an inviscid shear layer. Schade's (1964) approximate calculations indicate that the Landau constant is real and positive for the tanh profile which models this flow. Since the growth rate is complex, our remarks concerning two-dimensional non-linear waves should apply directly. Also it appears from observation that wake instability gives rise to non-linear travelling waves (vortex street) but calculations of the Landau constant are not yet available.

We conclude with several general comments. We note that the assumption $\mathrm{Re}\,\beta_c = O(1)$ is essential in our analysis. If, on the contrary, $\mathrm{Re}\,\beta_c = O(\epsilon)$ then the neglected terms containing fifth powers of the A_n's may be of the same size as the non-linear terms which are retained in (3.11). Furthermore, as noted under (5.13), our demonstration of consistency requires that $\mathrm{Re}\,\beta_c$ be $O(1)$.

In the Bénard and Taylor problems, β_c is real (positive) and $O(1)$ but the possibility that this is not always the case is raised by numerical calculations of the Landau constant for plane Poiseuille flow (Reynolds & Potter 1967 and Pekeris & Shkoller 1967; also see Nguyen & Davey 1970). These calculations show that $\mathrm{Re}\,\beta^{(k)}(R)$ changes sign near (k_c, R_c). Speaking generally, such a sign change may sharply restrict the range of Reynolds numbers for which our analysis could be expected to be uniformly valid in time. If $\mathrm{Re}\,\beta_c < 0$, as the above calculations show to be the case for plane Poiseuille flow, our analysis will be valid for a time but disturbances will eventually grow so large that the neglected quintic terms in the A_n's will be important.

As a second general comment, we observe that it is tempting to try to learn more about the infinite system of amplitude equations (3.11) by considering a truncated system. To do this for some fixed $R > R_c$ (fixed ϵ), one would restrict consideration to the modes $n = 0, \pm 1, \pm 2, \ldots, \pm M$ for some fixed positive integer M and replace the sum in (3.11) with a sum from $-M$ to M. It is essential that the integer M and the spacing Δk between successive modes be chosen so that not only all modes which grow according to linear theory, but also those which decay slowly, are included.

It would be interesting to do some numerical experimentation with such equations to see, for example, whether wave-number selection appears to be taking place. It should be borne in mind, however, that we have demonstrated the persistence of energy concentration in the most dangerous modes only asymptotically as $\epsilon \to 0$. For any fixed ϵ, energy will probably gradually leak to modes having wave-numbers farther and farther from k_0, so a truncation will presumably provide results to a given accuracy only for a limited period of time. The smaller ϵ and Δk, the longer this time should be.

One cannot help but be struck by the fundamental differences between the cases $\nu = 0$ (the Taylor problem and the Bénard problem) and the case $\nu \neq 0$ (the plane Poiseuille problem). First there is the difference in the governing equations. For $\nu = 0$ we see from either (5.11), or (5.30) with $\nu_1 = \nu_2 = 0$, that Z satisfies a diffusion type equation. Also the scaling $\tau = \epsilon^2 t$ eliminates ϵ from the equation and all terms are retained in the limit $\epsilon \to 0$ with τ and ω fixed. On the other hand, for $\nu \neq 0$, the first derivative term $\partial Z^*/\partial\omega$ is retained in (5.30). Depending on the scaling of ω and t different terms are lost in the limit $\epsilon \to 0$. In particular for the distinguished limit $\epsilon \to 0$ with $\tau = \epsilon^2 t$ and $\omega^* = \epsilon^2\xi$ fixed we obtain a wave type equation for the approximation Z_1^*.

We note that for $\nu = 0$ significant interaction between the most dangerous modes is obtained when $\Delta k = O(\epsilon)$, while for $\nu \neq 0$ one must take $\Delta k = O(\epsilon^2)$ to obtain the significant interactions (presence of ξ derivatives in the limiting equation). Since $\Delta k = O(\epsilon^2)$, for example, includes the possibility $\Delta k = o(\epsilon^2)$, decreasing Δk below a certain magnitude (an $O(\epsilon)$ magnitude when $\nu = 0$, an $O(\epsilon^2)$ magnitude when $\nu \neq 0$) yields no further significant interactions in the sense that the appropriate equations for the transforms (5.11) and (5.30) continue to describe the leading effects of interaction. Thus we expect these equations to be valid in the limit $\Delta k \to 0$ wherein the 'spectrum becomes continuous'. Of course the equations for the transforms emerge if we begin by assuming that the disturbance has the form of a modulated spatial oscillation, as in equations (5.16) and (5.29).

To summarize our analysis we have shown that, under certain widely met conditions, (i) initial energy concentrated in the most dangerous modes within $O(\epsilon)$ of k_c stays concentrated in these modes, (ii) a discrete (modal) analysis and a continuous (slowly varying) approach to non-linear stability problems are in harmony, (iii) an analysis of the development of a rather general initial disturbance for a large class of stability problems reduces to the study of a single canonical partial differential equation which involves only two constants of the basic system, (iv) this equation represents appropriate generalizations of the classical Landau equation for the growth or decay of a spatially periodic disturbance, (v) if the real part of the Landau constant is positive then the lowest-order non-linear terms in the multi-modal problem are stabilizing, (vi) for the case $\operatorname{Im}\mu_0^{(n)} = 0$ and β_c real and positive there exist steady multi-modal solutions, though whether there are stable solutions remains open.

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Appendix A

In this appendix we sketch the derivation of the fundamental system of amplitude equations (3.11). Substitution in (3.3) of the expressions given in (3.6) for the $\mathbf{\Phi}_k$ yields the following.

$k = 0$:

$$\left(\mathscr{L}_0 - \frac{\partial}{\partial t} S_0\right) \mathbf{\Psi}_0 = \sum_{l=1}^{N} \sum_{k \in J} [(\bar{P}_{k'}^{(l)} \bar{\mathbf{\Psi}}_{k'} . K_{k'}^{(l)}) Q_{k'}^{(l)} \mathbf{\Psi}_{k'} + (P_{k'}^{(l)} \mathbf{\Psi}_{k'} . \bar{K}_{k'}^{(l)}) \bar{Q}_{k'}^{(l)} \bar{\mathbf{\Psi}}_{k'}] + O(\delta_{k''}^2) + O(\epsilon^2), \quad \text{(A 1)}$$

where $k'' \in J^*$ and $\sum_{k \in J}$ means a sum over all the wave-numbers in J.

$k \in I$, $k \neq 0$:

$$\left(\mathscr{L}_k - \frac{\partial}{\partial t} S_k\right) \mathbf{\Psi}_k = \sum_{l=1}^{N} \sum_{k' \in J} [(\bar{P}_{k'}^{(l)} \mathbf{\Psi}_{k'} . K_{k+k'}^{(l)}) Q_{k+k'}^{(l)} \mathbf{\Psi}_{k+k'} + (P_{k+k'}^{(l)} \mathbf{\Psi}_{k+k'} . \bar{K}_{k'}^{(l)}) \bar{Q}_{k'}^{(l)} \bar{\mathbf{\Psi}}_{k'}] + O(\delta_{k''} \delta_{k+k''}) + O(\epsilon^2), \quad \text{(A 2)}$$

where $k'' \in J^*$.

$k \in Y$:

$$\left(\mathscr{L}_k - \frac{\partial}{\partial t} S_k\right) \mathbf{\Psi}_k = \sum_{l=1}^{N} \sum_{\substack{k' \in J \\ k' < k}} (P_{k'}^{(l)} \mathbf{\Psi}_{k'} . K_{k-k'}^{(l)}) Q_{k-k'}^{(l)} \mathbf{\Psi}_{k-k'} + O(\delta_{k''} \delta_{k-k''}) + O(\epsilon^2), \quad \text{(A 3)}$$

where $k'' \in J^*$.

$k \in J$:

$$\left(\mathscr{L}_k - \frac{\partial}{\partial t} S_k\right) \mathbf{\Psi}_k = \epsilon^2 \sum_{l=1}^{N} (P_0^{(l)} \mathbf{\Psi}_0 . K_k^{(l)}) Q_k^{(l)} \mathbf{\Psi}_k + (P_k^{(l)} \mathbf{\Psi}_k . K_0^{(l)}) Q_0^{(l)} \mathbf{\Psi}_0 + \Big(\sum_{\substack{k' \in J \\ k' < k}} + \sum_{\substack{k-k' \in J \\ k-k' < k}} \Big) [(P_{k'}^{(l)} \mathbf{\Psi}_{k'} . K_{k-k'}^{(l)}) Q_{k-k'}^{(l)} \mathbf{\Psi}_{k-k'}$$

$$+\Big(\sum_{\substack{k+k'\in J\\ k+k'>k}}+\sum_{k'\in J}\Big)(\bar{P}_{k'}\bar{\boldsymbol{\Psi}}_{k'}.K^{(l)}_{k+k'})\,Q^{(l)}_{k+k'}\boldsymbol{\Psi}_{k+k'}+(P^{(l)}_{k+k'}\boldsymbol{\Psi}_{k+k'}.\bar{K}^{(l)}_{k'})\,\bar{Q}^{(l)}_{k'}\bar{\boldsymbol{\Psi}}_{k'}]\}$$
$$+\epsilon^2O(\delta_{k''}\delta_{k-k''})+\epsilon^2O(\delta_{k''}\delta_{k+k''})+O(\epsilon^4)$$
$$=\epsilon^2\mathbf{B}_k+\epsilon^2O(\delta_{k''}\delta_{k-k''})+\epsilon^2O(\delta_{k''}\delta_{k+k''})+O(\epsilon^4), \quad \text{(A 4)}$$

where $k''\in I^*$.

Consider the error estimates in (A 1)–(A 4) that involve the δ_k's. Because of the rapid decay of the δ_k's for k in one of the sets J^*, I^*, Y^* (as shown in §4) these terms are negligible as $\epsilon\to 0$, and will not be written in the future.

Once the family of Fourier components $\boldsymbol{\Psi}_k$ for $k\in J$ is known, then the non-homogeneous terms in (A 1), (A 2) and (A 3) are known and we can solve for $\boldsymbol{\Psi}_0$, $\boldsymbol{\Psi}_k$ for $k\in I$ but $k\neq 0$, and $\boldsymbol{\Psi}_k$ for $k\in Y$. Let us consider each set of equations starting with the fundamental family of Fourier components for $k\in J$.

$k\in J$. From the results of §2 we can write

$$\boldsymbol{\Psi}_k(\eta,t)=\sum_{p=0}^{\infty}A_p^{(k)}(t)\,\boldsymbol{\phi}_p^{(k)}(\eta). \quad \text{(A 5)}$$

Substituting for $\boldsymbol{\Psi}_k(\eta,t)$ in (A 4) and taking the inner product with $\tilde{\boldsymbol{\phi}}_p^{(k)}(\eta)$ we find, omitting terms $O(\epsilon^4)$,

$$dA_p^{(k)}/dt+\mu_p^{(k)}A_p^{(k)}=-\epsilon^2(\mathbf{B}_k,\tilde{\boldsymbol{\phi}}_p^{(k)})$$
$$=-\epsilon^2b_p^{(k)}. \quad \text{(A 6)}$$

Recall that $\tau_0^{(k)}=O(\epsilon^2)$ if $k\in J$. Because $\mu_0^{(k)}$ is simple it follows that $\tau_p^{(k)}=O(1)$ for $p\geqslant 1$. Hence, from (A 6), $A_p^{(k)}=O(\epsilon^2)$ for $p\geqslant 1$ and

$$dA_0^{(k)}/dt+\mu_0^{(k)}A_0^{(k)}=-\epsilon^2b_0^{(k)},$$

so

$$d|A_0^{(k)}|^2/dt+2\tau_0^{(k)}|A_0^{(k)}|^2=-\epsilon^2[A_0^{(k)}\bar{b}_0^{(k)}+\bar{A}_0^{(k)}b_0^{(k)}]. \quad \text{(A 7)}$$

Further, it then follows from (A 5) that

$$\boldsymbol{\Psi}_k(\eta,t)=A_0^{(k)}(t)\,\boldsymbol{\phi}_0^{(k)}(\eta)+O(\epsilon^2)\quad (k\in J). \quad \text{(A 8)}$$

We can now exploit this result to simplify the equations for the forced terms $\boldsymbol{\Psi}_k$. We consider in turn wave-numbers k such that $k=0$, $k\in I$ but $k\neq 0$ and $k\in Y$.

$k=0$. Substituting for $\boldsymbol{\Psi}_k(\eta,t)$ from (A 8) in (A 1) and rearranging terms we obtain

$$\left(\mathscr{L}_0-\frac{\partial}{\partial t}S_0\right)\boldsymbol{\Psi}_0=\sum_{k'\in J}|A_0^{(k')}(t)|^2\,\mathbf{f}_{k'}(\eta)+O(\epsilon^2), \quad \text{(A 9)}$$

where

$$\mathbf{f}_{k'}(\eta)=\sum_{l=1}^{N}[(\bar{P}^{(l)}_{k'}\bar{\boldsymbol{\phi}}_0^{(k')}.K^{(l)}_{k'})\,Q^{(l)}_{k'}\boldsymbol{\phi}_0^{(k')}+(P_{k'}\boldsymbol{\phi}_0^{(k')}.\bar{K}^{(l)}_{k'})\,\bar{Q}^{(l)}_{k'}\bar{\boldsymbol{\phi}}_0^{(k')}] \quad \text{(A 10)}$$

is known. To solve (A 9) we write

$$\boldsymbol{\Psi}_0(\eta,t)=\sum_{p=0}^{\infty}A_p^{(0)}(t)\,\boldsymbol{\phi}_p^{(0)}(\eta). \quad \text{(A 11)}$$

Substituting for $\boldsymbol{\Psi}_0$ from (A 11) in (A 9) and taking the inner product with $\tilde{\boldsymbol{\phi}}_p^{(0)}(\eta)$ and omitting terms $O(\epsilon^2)$, we obtain

$$dA_p^{(0)}/dt+\mu_p^{(0)}A_p^{(0)}=-\sum_{k'\in J}|A_0^{(k')}(t)|^2\,(\mathbf{f}_{k'},\tilde{\boldsymbol{\phi}}_p^{(0)}). \quad \text{(A 12)}$$

Hence

$$A_p^{(0)}(t)=-e^{-\mu_p^{(0)}t}\left[\int_0^t e^{\mu_p^{(0)}t}\sum_{k'\in J}|A_0^{(k')}(t)|^2\,(\mathbf{f}_{k'},\tilde{\boldsymbol{\phi}}_p^{(0)})\,dt+C_p^{(0)}\right], \quad \text{(A 13)}$$

where $C_p^{(0)}$, the initial value of $A_p^{(0)}(t)$, is necessarily $O(1)$. Integrating by parts, and making use of the fact that $d|A_0^{(k')}(t)|^2/dt = O(\epsilon^2)$ for $k' \in J$ from (A 7),† we find

$$A_p^{(0)}(t) = -\sum_{k' \in J} \frac{|A_0^{(k')}(t)|^2}{\mu_p^{(0)}} (\mathbf{f}_{k'}, \tilde{\boldsymbol{\phi}}_0^{(k')}). \tag{A 14}$$

Consequently, from (A 11)

$$\boldsymbol{\Psi}_0(\eta, t) = \sum_{k' \in J} |A_0^{(k')}(t)|^2 \,\mathbf{g}_{k'}(\eta) + O(\epsilon^2), \tag{A 15}$$

where

$$\mathbf{g}_{k'}(\eta) = -\sum_{p=0}^{\infty} \frac{(\mathbf{f}_{k'}, \tilde{\boldsymbol{\phi}}_p^{(0)})}{\mu_p^{(0)}} \boldsymbol{\phi}_p^{(0)}(\eta). \tag{A 16}$$

Once the $A_0^{(k')}(t)$ are known, (A 15) gives the distortion of the mean motion.

Alternatively, and more simply from a computational point of view, we can look for a solution of (A 9) of the form (A 15). Making use of the fact that $d|A_0^{(k')}(t)|^2/dt = O(\epsilon^2)$ we obtain the equation

$$\mathscr{L}_0 \mathbf{g}_{k'} = \mathbf{f}_{k'}, \tag{A 17}$$

subject to appropriate homogeneous boundary conditions.

$k \in I$, $k \neq 0$: Proceeding in the same manner as for the case $k = 0$ we find that

$$\boldsymbol{\Psi}_k(\eta, t) = \sum_{k' \in J} \bar{A}_0^{(k')}(t)\, A_0^{(k+k')}(t)\, \mathbf{g}_{k', k+k'}(\eta) + O(\epsilon^2) \quad (k \in I,\, k \neq 0), \tag{A 18}$$

where

$$\{\mathscr{L}_k - i(\nu_0^{(k')} - \nu_0^{(k+k')})\, S_k\}\, \mathbf{g}_{k', k+k'} = \mathbf{f}_{k', k+k'}. \tag{A 19}$$

Here $\nu_0^{(k')} = \mathrm{Im}\,(\mu_0^{(k')})$, and

$$\mathbf{f}_{k', k+k'}(\eta) = \sum_{l=1}^{N} [(\bar{P}_{k'}^{(l)} \bar{\boldsymbol{\phi}}_0^{(k')} . K_{k+k'}^{(l)})\, Q_{k+k'}^{(l)} \boldsymbol{\phi}_0^{(k+k')} + (P_{k+k'} \boldsymbol{\phi}_0^{(k+k')} . K_{k'}^{(l)})\, Q_{k'}^{(l)} \boldsymbol{\phi}_0^{(k')}]. \tag{A 20}$$

Appropriate homogeneous boundary conditions on $\mathbf{g}_{k', k+k'}$ are specified at $\eta = 0$ and $\eta = 1$.

$k \in Y$: Again proceeding in the same manner as for the case $k = 0$ we find that

$$\boldsymbol{\Psi}_k(\eta, t) = \sum_{\substack{k' \in J \\ k' < k}} A_0^{(k')}(t)\, A_0^{(k-k')}(t)\, \mathbf{g}_{k', k-k'}(\eta) + O(\epsilon^2) \quad (k \in Y), \tag{A 21}$$

where

$$\{\mathscr{L}_k + i(\nu_0^{(k')} + \nu_0^{(k-k')})\, S_k\}\, \mathbf{g}_{k', k-k'}(\eta) = \mathbf{f}_{k', k-k'}(\eta), \tag{A 22}$$

and

$$\mathbf{f}_{k', k-k'}(\eta) = \sum_{l=1}^{N} (P_{k'}^{(l)} \boldsymbol{\phi}_0^{(k')} . K_{k-k'}^{(l)})\, Q_{k-k'}^{(l)} \boldsymbol{\phi}_0^{(k-k')}, \tag{A 23}$$

and appropriate homogeneous boundary conditions on $\mathbf{g}_{k', k-k'}$ are specified at $\eta = 0$ and $\eta = 1$.

It is clear from these results that for $k \in I$, Y, $\boldsymbol{\Psi}_k(\eta, t) = O(1)$ if $A_0^{(k)}(t) = O(1)$ for $k \in J$.

Now that $\boldsymbol{\Psi}_0$, $\boldsymbol{\Psi}_k$ for $k \in I$ but $k \neq 0$, and $\boldsymbol{\Psi}_k$ for $k \in Y$ are known in terms of $A_0^{(k)}(t)$ it is possible to (*a*) compute $\mathbf{B}_k$ given in (A 4), (*b*) compute $b_0^{(k)}$ given in

† Though asymptotically correct as $\epsilon \to 0$, caution must be used in practice for the case of plane Poiseuille flow since the neglected term is also multiplied by the Reynolds number which is large. This was pointed out to us in a private communication by A. Davey; see Nguyen & Davey (1970).

(A 6), and hence know the terms $b_0^{(k)}$ and $\bar{b}_0^{(k)}$ in (A 7) for $A_0^{(k)}(t)$, (*c*) solve (A 7) for the unknown function $|A_0^{(k)}(t)|$. The calculation of $\mathbf{B}_k$ is a formidable algebraic task. The result of this calculation is that $b_0^{(k)} = (\mathbf{B}_k, \tilde{\boldsymbol{\phi}}_0^{(k)})$ is given by

$$\begin{aligned} b_0^{(k)} = {} & A_0^{(k)}(t) \sum_{k' \in J} |A_0^{(k')}(t)|^2 C^{(1)}(k,k') + \sum_{\substack{k-k' \in J \\ k-k'<k}} A_0^{(k-k')} \sum_{k'' \in J} \bar{A}_0^{(k'')} A_0^{(k'+k'')} C^{(2)}(k,k',k'') \\ & + \sum_{\substack{k' \in J \\ k'<J}} A_0^{(k')} \sum_{k'' \in J} \bar{A}_0^{(k'')} A_0^{(k-k'+k'')} C^{(3)}(k,k',k'') \\ & + \sum_{\substack{k+k' \in J \\ k+k'>k}} A_0^{(k+k')} \sum_{k'' \in J} A_0^{(k'')} \bar{A}_0^{(k'+k'')} C^{(4)}(k,k',k'') \\ & + \sum_{k' \in J} \bar{A}_0^{(k')} \sum_{k'' \in J} A_0^{(k'')} A_0^{(k+k'-k'')} C^{(5)}(k,k',k'') + O(\epsilon^2) \quad (k \in J). \end{aligned} \qquad \text{(A 24)}$$

The functions $C^{(1)}, C^{(2)}, \ldots, C^{(5)}$ are given in appendix B.

The saving feature of the calculation is that we need only evaluate the coefficients $C^{(1)}, C^{(2)}, \ldots, C^{(5)}$ in the limit $\epsilon \to 0$. This is because the term on the right-hand side of (A 7) has a factor ϵ^2; hence terms $O(\epsilon)$ in $b_0^{(k)}$ will yield terms $O(\epsilon^3)$ in (A 7), and these terms can be neglected. The limits of $C^{(1)}, C^{(2)}, \ldots, C^{(5)}$ as $\epsilon \to 0$ are evaluated in appendix B; they are denoted by $c^{(1)}, \ldots, c^{(5)}$ respectively. As discussed at the beginning of §3, the assumption $|k_0 - k_c| = O(\epsilon^2)$ means that $k_0 \to k_c$ as $\epsilon \to 0$. This means that in evaluating the limits as $\epsilon \to 0$, if $k' \in J$ then $k' \to k_c$, if $k' \in I$ then $k' \to 0$, and if $k' \in Y$ then $k' \to 2k_c$. The constants $c^{(1)}, \ldots, c^{(5)}$ are given by

$$\left. \begin{aligned} c^{(1)} &= \sum_{l=1}^{N} ((P_0^{(l)} \mathbf{g}_0 . K_{k_c}^{(l)}) Q_{k_c}^{(l)} \boldsymbol{\phi}_0^{(k_c)} + (P_{k_c}^{(l)} \boldsymbol{\phi}_0^{(k_c)} . K_0^{(l)}) Q_0^{(l)} \mathbf{g}_0, \tilde{\boldsymbol{\phi}}_0^{(k_c)}), \\ c^{(2)} &= \sum_{l=1}^{N} ((P_0^{(l)} \mathbf{g}_0 . K_{k_c}^{(l)}) Q_{k_c}^{(l)} \boldsymbol{\phi}_0^{(k_c)}, \tilde{\boldsymbol{\phi}}_0^{(k_c)}), \\ c^{(3)} &= \sum_{l=1}^{N} ((P_{k_c}^{(l)} \boldsymbol{\phi}_0^{(k_c)} . K_0^{(l)}) Q_0^{(l)} \mathbf{g}_0, \tilde{\boldsymbol{\phi}}_0^{(k_c)}), \\ c^{(4)} &= c^{(1)}, \\ c^{(5)} &= \sum_{l=1}^{N} ((\bar{P}_{k_0}^{(l)} \bar{\boldsymbol{\phi}}_0^{(k_c)} K_{2k_c}^{(l)}) Q_{2k_c}^{(l)} \mathbf{g}_2 + (P_{2k_c}^{(l)} \mathbf{g}_2 . \bar{K}_{k_c}^{(l)}) \bar{Q}_{k_c}^{(l)} \bar{\boldsymbol{\phi}}_0^{(k_c)}, \tilde{\boldsymbol{\phi}}_0^{(k_c)}), \end{aligned} \right\} \qquad \text{(A 25)}$$

where $\mathbf{g}_0(\eta)$ and $\mathbf{g}_2(\eta)$ are solutions of

$$\mathscr{L}_0 \mathbf{g}_0 = \mathbf{f}_0, \quad (\mathscr{L}_{2k_c} + i2\nu^{(2k_c)} S_{2k_c}) \mathbf{g}_2 = \mathbf{f}_2, \qquad \text{(A 26)}$$

with the appropriate boundary conditions, and

$$\left. \begin{aligned} \mathbf{f}_0(\eta) &= \sum_{l=1}^{N} [(\bar{P}_{k_c} \bar{\boldsymbol{\phi}}_0^{(k_c)} . K_{k_c}^{(l)}) Q_{k_c}^{(l)} \boldsymbol{\phi}_0^{(k_c)} + (P_{k_c}^{(l)} \boldsymbol{\phi}_0^{(k_c)} . \bar{K}_{k_c}^{(l)}) \bar{Q}_{k_c}^{(l)} \bar{\boldsymbol{\phi}}_0^{(k_c)}], \\ \mathbf{f}_2(\eta) &= \sum_{l=1}^{N} (P_{k_c}^{(l)} \boldsymbol{\phi}_0^{(k_c)} . K_{k_c}^{(l)}) Q_{k_c}^{(l)} \boldsymbol{\phi}_0^{(k_c)}. \end{aligned} \right\} \qquad \text{(A 27)}$$

(Instead of evaluating the functions in the above formulas at k_c, we could have evaluated them at k_0, for the difference in the final solution is negligible to the order to which we carry the calculations. It is advantageous, however, to use the same values for $c^{(1)}, \ldots, c^{(5)}$ regardless of the choice of k_0.)

In terms of the notation of (3.10a), (A 24) reduces to

$$b_0^{(n)} = c^{(1)}A_n \sum_{p\in J} |A_p|^2 + c^{(2)} \sum_{\substack{p\in J\\ p\geqslant 1}} A_{n-p} \sum_{m\in J} \bar{A}_m A_{m+p} + c^{(3)} \sum_{\substack{p\in J\\ p<n}} A_p \sum_{m\in J} \bar{A}_m A_{n-p+m}$$
$$+ c^{(4)} \sum_{\substack{p\in J\\ p\geqslant 1}} A_{n+p} \sum_{m\in J} A_m \bar{A}_{p+m} + c^{(5)} \sum_{m\in J} \bar{A}_m \sum_{p\in J} A_p A_{p+n-m} + O(\epsilon). \quad \text{(A 28)}$$

where $b_0^{(k)} = b_0^{(k_0+n\sigma\epsilon)} = b_0^{(n)}$ for $k\in J$.

It is possible to simplify (A 28) by noting that the sums multiplying $c^{(2)}$, $c^{(4)}$ and $c^{(5)}$ in (A 5) can be written as follows:

$$\left.\begin{aligned} \sum_{\substack{p\in J\\ p\geqslant 1}} A_{n-p} \sum_{m\in J} \bar{A}_m A_{m+p} &= \sum_{\substack{p\in J\\ p<n}} A_p \sum_{m\in J} \bar{A}_m A_{n-p+m}, \\ \sum_{\substack{p\in J\\ p\geqslant 1}} A_{n+p} \sum_{m\in J} A_m \bar{A}_{p+m} &= \sum_{\substack{p\in J\\ p\geqslant n}} A_p \sum_{m\in J} \bar{A}_m A_{m-p+n}, \\ \sum_{m\in J} \bar{A}_m \sum_{p\in J} A_p A_{p+n-m} &= \sum_{\substack{p\in J\\ p<n}} A_p \sum_{m\in J} \bar{A}_m A_{n+m-p} \\ &\quad + A_n \sum_{m\in J} |A_m|^2 + \sum_{\substack{p\in J\\ p>n}} A_p \sum_{m\in J} \bar{A}_m A_{n+m-p}. \end{aligned}\right\} \quad \text{(A 29)}$$

We next observe that

$$c^{(1)} = c^{(2)} + c^{(3)}, \quad c^{(4)} = c^{(1)}, \quad c^{(1)} + c^{(5)} = \beta^{(k_c)}.$$

We make the abbreviation $\beta^{(k_c)} = \beta_c$ and note that β_c is the Landau constant associated with a wave-number k_c. Using the above results and substituting the sums of (A 29) in (A 28) we find for $k\in J$ that

$$b_0^{(n)} = \beta_c [A_n \sum_{m\in J} |A_m|^2 + \sum_{\substack{p\in J\\ p<n}} A_p \sum_{m\in J} \bar{A}_m A_{n-p+m} + \sum_{\substack{p\in J\\ p>n}} A_p \sum_{m\in J} \bar{A}_m A_{n-p+m}]$$
$$= \beta_c \sum_{p\in J} A_p \sum_{m\in J} \bar{A}_m A_{n+m-p} + O(\epsilon). \quad \text{(A 30)}$$

Finally, substituting for $b_0^{(k)}$ in (3.9), we obtain (3.11).

Appendix B

The functions $C^{(1)}, \ldots, C^{(5)}$ are given by

$$C^{(1)}(k,k') = \sum_{l=1}^{N} ((P_0^{(l)} \mathbf{g}_{k'} . K_k^{(l)}) Q_k^{(l)} \boldsymbol{\phi}_0^{(k)} + (P_k^{(l)} \boldsymbol{\phi}_0^{(k)} . K_0^{(l)}) Q_0^{(l)} \mathbf{g}_{k'}, \tilde{\boldsymbol{\phi}}_0^{(k)}), \quad \text{(B 1)}$$

where $k\in J$ and $k'\in J$ and $\mathbf{g}_{k'}$ is the solution of (A 17).

$$C^{(2)}(k,k',k'') = \sum_{l=1}^{N} ((P_{k'}^{(l)} \mathbf{g}_{k'',k'+k''} . K_{k-k'}^{(l)}) Q_{k-k'}^{(l)} \boldsymbol{\phi}_0^{(k-k')}, \tilde{\boldsymbol{\phi}}_0^{(k)}), \quad \text{(B 2)}$$

where $k\in J$, $k'\in I$, $k''\in J$ and $\mathbf{g}_{k'',k'+k''}$ is the solution of (A 19).

$$C^{(3)}(k,k',k'') = \sum_{l=1}^{N} ((P_{k'}^{(l)} \boldsymbol{\phi}_0^{(k')} . K_{k-k'}^{(l)}) Q_{k-k'}^{(l)} \mathbf{g}_{k'',k-k'+k''}, \tilde{\boldsymbol{\phi}}_0^{(k)}), \quad \text{(B 3)}$$

where $k\in J$, $k'\in J$, $k''\in J$ and $\mathbf{g}_{k'',k'+k''}$ is the solution of (A19).

$$C^{(4)}(k,k',k'') = \sum_{l=1}^{N} ((\bar{P}^{(l)}_{k'}\bar{\mathbf{g}}_{k'',k'+k''}.K^{(l)}_{k+k'})\,Q^{(l)}_{k+k'}\,\boldsymbol{\phi}_0^{(k+k')} + (P^{(l)}_{k+k'}\,\boldsymbol{\phi}_0^{(k+k')}.\bar{K}^{(l)}_{k'})\,\bar{Q}^{(l)}_{k'}\bar{\mathbf{g}}_{k'',k'+k''},\tilde{\boldsymbol{\phi}}_0^{(k)}), \qquad \text{(B 4)}$$

where $k \in J$, $k' \in I$, $k'' \in J$ and $\mathbf{g}_{k'',k'+k''}$ is the solution of (A 19).

$$C^{(5)}(k,k',k'') = \sum_{l=1}^{N} ((\bar{P}^{(l)}_{k'}\bar{\boldsymbol{\phi}}_0^{(k')}.K^{(l)}_{k+k'})\,Q^{(l)}_{k+k'}\,\mathbf{g}_{k'',k+k'-k''} + (P^{(l)}_{k+k'}\,\mathbf{g}_{k'',k+k'-k''}.\bar{K}^{(l)}_{k'})\,\bar{Q}^{(l)}_{k'}\bar{\boldsymbol{\phi}}_0^{(k)},\tilde{\boldsymbol{\phi}}_0^{(k)}), \qquad \text{(B 5)}$$

where $k \in J$, $k' \in J$, $k'' \in J$ and $\mathbf{g}_{k'',k+k'-k''}$ is the solution of (A 22).

To evaluate the functions $C^{(1)}, \ldots, C^{(5)}$ first recall that as $\epsilon \to 0$ we have if $k' \in J$ then $k' \to k_c$, if $k' \in I$ then $k' \to 0$, and if $k' \in J$ then $k' \to 2k_c$. Consider (B 1). Since $k' \in J$ and $k \in J$ it follows that

$$\boldsymbol{\phi}_0^{(k)} \to \boldsymbol{\phi}_0^{(k_c)}, \quad \tilde{\boldsymbol{\phi}}_0^{(k)} \to \tilde{\boldsymbol{\phi}}_0^{(k_c)}, \quad \mathbf{g}_{k'} \to \mathbf{g}_{k_c}, \qquad \text{(B 6)}$$

as $\epsilon \to 0$, where from (A 17)

$$\mathscr{L}_0\,\mathbf{g}_{k_c} = \mathbf{f}_{k_c}, \qquad \text{(B 7)}$$

and from (A 10)

$$\mathbf{f}_{k_c}(\eta) = \sum_{l=1}^{N} [(\bar{P}^{(l)}_{k_c}\bar{\boldsymbol{\phi}}_0^{(k_c)}.K^{(l)}_{k_c})\,Q^{(l)}_{k_c}\,\boldsymbol{\phi}_0^{(k_c)} + (P^{(l)}_{k_c}\,\boldsymbol{\phi}_0^{(k_c)}.\bar{K}^{(l)}_{k_c})\,\bar{Q}^{(l)}_{k_c}\,\boldsymbol{\phi}_0^{(k_c)}]. \qquad \text{(B 8)}$$

For convenience we denote $\mathbf{g}_{k_c}$ by $\mathbf{g}_0$ and $\mathbf{f}_{k_c}$ by $\mathbf{f}_0$. Thus $c^{(1)} = \lim_{\epsilon\to 0} C^{(1)}(k,k')$ is given by (B 1) with $\boldsymbol{\phi}_0^{(k)}$, $\tilde{\boldsymbol{\phi}}_0^{(k)}$, and $\mathbf{g}_{k'}$ replaced by $\boldsymbol{\phi}_0^{(k_c)}$, $\tilde{\boldsymbol{\phi}}_0^{(k_c)}$, and $\mathbf{g}_0$. The results are recorded in (A 25)–(A 27).

Next consider (B 2). Since $k \in J$, $k' \in I$, $k'' \in J$ it follows that

$$\boldsymbol{\phi}_0^{(k-k')} \to \boldsymbol{\phi}_0^{(k_c)}, \quad \tilde{\boldsymbol{\phi}}_0^{(k)} \to \tilde{\boldsymbol{\phi}}_0^{(k_c)}, \quad \mathbf{g}_{k'',k'+k''} \to \mathbf{g}_{k_c,k_c}, \qquad \text{(B 9)}$$

as $\epsilon \to 0$, where from (A 19) with k replaced by k' and k' replaced by k''

$$\mathscr{L}_0\,\mathbf{g}_{k_c,k_c} = \mathbf{f}_{k_c,k_c}, \qquad \text{(B 10)}$$

and from (A 20)

$$\mathbf{f}_{k_c,k_c}(\eta) = \sum_{l=1}^{N} [(\bar{P}^{(l)}_{k_c}\bar{\boldsymbol{\phi}}_0^{(k_c)}.K^{(l)}_{k_c})\,Q^{(l)}_{k_c}\,\boldsymbol{\phi}_0^{(k_c)} + (P^{(l)}_{k_c}\,\boldsymbol{\phi}_0^{(k_c)}.K^{(l)}_{k_c})\,Q^{(l)}_{k_c}\,\boldsymbol{\phi}_0^{(k_c)}]. \qquad \text{(B 11)}$$

Comparison of (B 11) and (B 8) shows that

$$\mathbf{f}_{k_c,k_c}(\eta) = \mathbf{f}_{k_c}(\eta) = \mathbf{f}_0(\eta),$$

and hence from (B 10) and (B 7) it follows that

$$\mathbf{g}_{k_c,k_c}(\eta) = \mathbf{g}_{k_c}(\eta) = \mathbf{g}_0(\eta).$$

The result for $c^{(2)} = \lim_{\epsilon\to 0} C^{(2)}(k,k',k'')$ is recorded in (A 25). The evaluation of $c^{(3)}$ and $c^{(4)}$ is similar; the results are given in (A 25).

Finally consider (B 5). Since $k \in J$, $k' \in J$ and $k'' \in J$ it follows that

$$\boldsymbol{\phi}_0^{(k')} \to \boldsymbol{\phi}_0^{(k_c)}, \quad \boldsymbol{\phi}_0^{(k)} \to \boldsymbol{\phi}_0^{(k_c)}, \quad \mathbf{g}_{k'',k+k'-k''} \to \mathbf{g}_{k_c,k_c}, \qquad \text{(B 12)}$$

as $\epsilon \to 0$, where from (A 22) with k replaced by $k+k'$ and k' replaced by k''

$$(\mathscr{L}_{2k_c} + i2\nu_0^{(k_c)} S_{2k_c})\, \mathbf{g}_{k_c,k_c} = \mathbf{f}_{k_c,k_c}, \tag{B 13}$$

and from (A 23)

$$\mathbf{f}_{k_c,k_c}(\eta) = \sum_{l=1}^{N} (P_{k_c}^{(l)} \boldsymbol{\phi}_0^{(k_c)} . K_{k_c}^{(l)})\, Q_{k_c}^{(l)} \boldsymbol{\phi}_0^{(k_c)}. \tag{B 14}$$

For convenience we have denoted the $\mathbf{g}_{k_c,k_c}$ of (B 13) and $\mathbf{f}_{k_c,k_c}$ of (B 14) by $\mathbf{g}_2$ and $\mathbf{f}_2$ respectively. Thus $c^{(5)} = \lim_{\epsilon\to 0} C^{(5)}(k,k',k'')$ is given by (B 5) with $\boldsymbol{\phi}_0^{(k')}$, $\boldsymbol{\phi}_0^{(k)}$ and $\mathbf{g}_{k'',k+k'-k''}$ replaced by $\boldsymbol{\phi}_0^{(k_c)}$, $\boldsymbol{\phi}_0^{(k_c)}$ and $\mathbf{g}_2$ respectively. The results are recorded in (A 25)–(A 27).

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